3.1 Topological Spaces: Fundamentals

Theorem 3.1. Let $\{U_i\}_{i=1}^{n}$ be a finite collection of open sets in a topological space $(X, \mathcal{T})$. Then $\cap_{i=1}^{n} U_i$ is open.

Proof: We'll do this with mathematical induction. Let the base case be $n = 1$. Then observe that

$$\bigcap_{i=1}^{1} U_i = U_i$$

which is open by hypothesis.

Now we perform the inductive step. Suppose the statement holds for the integer $n \geq 1$. Let V be any open set. Then

$$\left(\bigcap_{i=1}^{n} U_i\right) \cap V$$

is the intersection of two open sets, (by hypothesis, both $\bigcap\limits_{i=1}^{n} U_i$ and V) are open, and so by condition (3) of a topology the intersection of $n+1$ open sets is open. As the statement holds for $n+1$, we have by mathematical induction that it holds for all $n \in \mathbb{N}$.

■

Exercise 3.2 Why does your proof not prove the false statement that the infinite intersection of open sets is necessarily open?

Solution: The answer to this lies in the fact that a proposition which is proven to be true by mathematical induction does not imply that the proposition is true for an infinite number of steps. Thus the proof does not prove the false statement that infinite intersections are open.

□

Theorem 3.3 A set U is open in a topological space $(X, \mathcal{T})$ if and only if for every point $x \in U$, there exists an open set U_x such that $x \in U_x \subset U$.

Proof: We'll prove one direction at a time. Suppose that we have a set U such that, for every $x \in U$, there exists an open set U_x such that $x \in U_x \subset U$. Now suppose we take the (possibly uncountable) union of each of these open sets U_x. Observe that, since for each x we have $U_x \subset U$,

$$\bigcup_{x \in U} U_x = U.$$

However by condition (4) in the definition of a topology, we know that this ought to be inside our topology $\mathcal{T}$, which proves that U must be an open set.

Now we prove the other direction. Consider an arbitrary point $x \in U$, where U is an open set in our topology $\mathcal{T}$. Let V be any neighborhood about x. Observe that $U \cap V$ is an open set such that $x \in U_x \subset U$. Thus we have found our open neighborhood U_x, proving the other direction of the theorem. Thus the theorem itself is true.

■

Exercise 3.4 Verify that $\mathcal{T}_{\text{std}}$ is a topology on $\mathbb{R}^n$; in other words, it satisfies the four conditions of the definition of a topology.

Solution:

1. Observe that the first condition is satisfied, namely that $\emptyset \in \mathcal{T}_{\text{std}}$. This is because the condition to be in $\mathcal{T}_{\text{std}}$ is vacuously true for the empty set because there are no elements in the empty set.

2. Now consider the set $\mathbb{R}^n$ itself. For any point $p \in \mathbb{R}^n$, $B(p, \epsilon) \subset \mathbb{R}^n$ for any $\epsilon > 0$. Thus by the definition of $\mathcal{T}_{\text{std}}$, we have that $\mathbb{R}^n \in \mathcal{T}_{\text{std}}$. Condition two is satisfied.

3. Now consider two elements $U, V \in \mathcal{T}_{\text{std}}$. Suppose that $U \cap V \neq \emptyset$; otherwise it is trivial. So consider an element $p \in U \cap V$. Then there exists two balls $B(p, \epsilon_1) \subset U$ and $B(p, \epsilon_2) \subset V$ where $\epsilon_1, \epsilon_2 > 0$. On this subset, observe that $B(p, \min\{\epsilon_1, \epsilon_2\}) \subset U \cap V$. First note that we can certainly conclude that $B(p, \epsilon_1) \cap B(p, \epsilon_2) \subset U \cap V$. Now because $B(p, \epsilon_1)$ and $B(p, \epsilon_2)$ are balls about the same point, we know that $B(p, \epsilon_1) \cap B(p, \epsilon_2) = B(p, \min\{\epsilon_1, \epsilon_2\})$, so that we may conclude $U \cap V \in \mathcal{T}_{\text{std}}$. Thus condition three is satisfied.

4. Finally, we'll verify the fourth condition. Consider $\{U_\alpha\}_{\alpha\in\lambda}$ where λ is an arbitrary index set such that $U_\alpha \in \mathcal{T}_{\text{std}}$. Thus for each $\alpha \in \lambda$, and for every point $p \in U_\alpha$, there exists an open ball $B(p, \epsilon_{(\alpha,p)}) \subset U_\alpha$ such that $\epsilon_{(\alpha,p)} > 0$. Next, suppose $p \in \bigcup\limits_{\alpha\in\lambda} U_\alpha$. Then $p \in U_\alpha$ for at least one $\alpha \in \lambda$, so that $B(p, \epsilon(\alpha, p)) \subset U_\alpha$. Since p was arbitrary in $\bigcup\limits_{\alpha\in\lambda} U_\alpha$, we have that $\bigcup\limits_{\alpha\in\lambda} U_\alpha \in \mathcal{T}_{\text{std}}$ as desired.

$\square$

3.5 Verify that the discrete, indiscrete, finite complement and countable complement topologies are indeed topologies on any set X.

Solution: We can verify that the finite complement topology $\mathcal{T}$ on a set X is a true topology on X as follows.

1. First observe that in the definition of the topology $\emptyset$ is said to be in the topology so the first condition of a topology is satisfied.

2. Next we can verify the second property of topologies. It is obvious that $X \in \mathcal{T}$. This is because $X - X = \emptyset$ which is itself a finite set.

3. Now if $U, V \in \mathcal{T}$, then $X - U$ and $X - V$ are both finite sets. Therefore, we can conclude that $(X - U) \cup (X - V)$ is a finite set. However, by De Morgan's laws, $(X - U) \cup (X - V) = X - (U \cap V)$, and because this is a finite set, we must conclude that $U \cap V \in \mathcal{T}$. Thus the third property of a topology is verified.

4. Finally, we verify the last property in the defintion of a topology. Suppose $U_\beta \in \mathcal{T}$ for all $\beta \in \lambda$. Now observe that for some $\beta \in \lambda$, $X - U_\beta$ is a finite set. But observe that $X - \cup_{\alpha\in\lambda} U_\alpha \subset X - U_\beta$, so that $X - \cup_{\alpha\in\lambda} U_\alpha$ must also be a finite set. Thus we see that $\cup_{\alpha\in\lambda} U_\alpha \in \mathcal{T}$, proving the last property which verifies that $\mathcal{T}$ is a true topology on X.

$\square$

Exercise 3.7 Give an example of a topological space and a collection of open sets in that topological space that show that infinite intersections of open sets need not be open

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Solution: We can borrow the example I provided in Exercise 3.2. Consider the standard topology on $\mathbb{R}$ and observe that $\bigcap_{n=1}^{\infty}\left(-\frac{1}{n},\frac{1}{n}\right) = \{0\}$. $\{0\}$ isn't an open set under the standard topology, so that this example shows that countable intersections of open sets may not be open.

$\square$

Exercise 3.8 Let $X = \mathbb{R}$ and $A = (1,2)$. Verify that 0 is a limit point A in the indiscrete topology and the finite complement topology, but not in the standard topology nor the discrete topology of $\mathbb{R}$.

Solution: In the indiscrete topology, the only possible set that can contain 0 is simply $\mathbb{R}$ itself, for which $\mathbb{R} \cap (1,2) \neq \emptyset$. Thus 0 must be a limit point of $(1,2)$.

In the finite complement topology on $\mathbb{R}$, the open sets which contain 0 must be sets U such that $\mathbb{R} - U$ is finite and $0 \in U$. Now since $\mathbb{R} - U$ must be finite while (1,2) is obviously uncountable, it will never be the case that $(1,2) \subset (\mathbb{R} - U)$. Therefore, $(U - \{0\}) \cap (1,2)) \neq \emptyset$ for all U in the finite complement topology. Thus 0 must be a limit point of $(1,2)$ in this toplogy.

Now 0 is obviously not a limit point of (1,2) in the standard topology. This can be demonstrated by simply constructing a ball such as $B(0,1/2)$ (a ball about 0 of radius 1/2) to show the existence of one open set U about 0 such that $U \cap A = \emptyset$. Hence, 0 is not a limit point of (1,2).

0 is also not a limit point of (1,2) in the discrete topology. For example, consider the open set $\{0\}$ which contains 0 but obviously $\{0\} \cap (1,2) = \emptyset$. Again, by theorem 3.9, we can see that 0 is not a limit point of (1,2) in this topology.

$\square$

Theorem 3.9 Suppose $p \notin A$ in a topological space $(X, \mathcal{T})$. Then p is not a limit point of A if and only if there exists a neighborhood U of p such that $U \cap A = \emptyset$.

Proof: First suppose that there exists a exists a neighborhood U of p such that $U \cap A = \emptyset$. Then by definition, this cannot be a limit point, since the requirement to be a limit point is that every neighborhood of p must contain a point $q \neq p$ where $q \in A$. Clearly we we see that this condition cannot be satisfied, so p cannot be a limit point.

We can prove the other direction by supposing now that p is not a limit point of U. Since p is not a limit point, we know by definition that there exists at least one neighborhood U of p such that $(U - \{p\}) \cap A = \emptyset$. Since we are given that $p \notin A$, we can further state that $U \cap A = \emptyset$. Thus we have found our set U of p such that $U \cap A = \emptyset$, which proves the theorem.

■

Exercise 3.10 If p is an isolated point of a set A in a topological space X, then there exists an open set U such that $U \cap A = \{p\}$.

Solution: Since p is an isolated point, we know that p is not a limit point of A. By definition of a limit point, this means that there exists at least one open set U containing p such that $(U - \{p\}) \cap A = \emptyset$. Since $p \in A$ and $p \in U$, we can then state that $U \cap A = \{p\}$, as desired. Thus such a U described in the problem statement exists.

□

Exercise 3.11 Give examples of sets A in various topological spaces $(X, \mathcal{T})$ with
1. A limit point of A that is an element A;
2. A limit point of A that is not an element of A;
3. An isolated point of A;
4. A point not in A that is not a limit point of A;

Solution:

1. Consider the standard topology $\mathcal{T}_{\text{std}}$ on $\mathbb{R}$. For any interval $(a, b) \subset \mathbb{R}$ where $a < b$ we have that any point $x \in (a, b)$ is a limit point since, for any neighborhood U about x, $(U - \{x\}) \cap (a, b) \neq \emptyset$. since for any neighborhood of x there exists a ball $B(x, \epsilon)$ such that $B(x, \epsilon) \subset U$.

2. For any interval (a, b) as defined in (1.), we have that a and b are both limit points of the interval. This is because any open set about these two points will always include other points in (a, b). For example, if we construct a ball $B(a, \epsilon)$ (a neighborhood about a with radius ϵ) then any point in the interval $(a, a+\epsilon) \subset (a, b)$ can be found within the ball, so that $B(p, \epsilon) \cap (a, b) \neq \emptyset$. This analogously holds for b, so for any open set U about a or b, we have that $(U - \{a\}) \cap (a, b) \neq \emptyset$ or $(U - \{b\}) \cap (a, b) \neq \emptyset$, so that a, b are both limit points of (a, b).

3. Let $x \in \mathbb{R}$ such that $x \notin (a, b)$, and observe that x is an isolated point of the set $\{x\} \cup (a, b)$. In this example, x is quite literally an isolated point!

4. Any point $x \notin (a, b)$ is a point that is not in (a, b) and is not a limit point of (a, b).

□

Theorem 3.13 For any topological space $(X, \mathcal{T})$ and $A \subset X$, $\overline{A}$ is closed. That is, for any set A in a topological space, $\overline{\overline{A}} = \overline{A}$,

Proof: To prove this, let p be a limit point of $\overline{A}$. Then for every open set U which contains p, we know that

$$(U - \{p\}) \cap \overline{A} \neq \emptyset.$$

Thus for each U there exists a point $q \in \overline{A}$ such that $q \in U$ and $q \neq p$.
If $q \in A$, then we see that

$$(U - \{p\}) \cap A \neq \emptyset.$$

If q is a limit point of A, then every open set containing q must intersect with A. Since $U - \{p\}$ is an open set containing q, we can also conclude that the set $U - \{p\}$ must itself intersect with A. Either way, we have shown that for every open set U which contains p, $(U - \{p\}) \cap A \neq \emptyset$. In other words, if p is a limit point of $\overline{A}$ then $p \in \overline{A}$, so $\overline{\overline{A}} \subset \overline{A}$. Since it is trivial that $\overline{A} \subset \overline{\overline{A}}$, we must have that $\overline{\overline{A}} = \overline{A}$ as desired.

■

Theorem 3.14 Let $(X, \mathcal{T})$ be a topological space. Then the set A is closed if and only if $X - A$ is open.

Proof: First we begin with the forward direction by supposing A is a closed set. Then A must contain all of its limit points, so $X - A$ contains no limit points of A.
By Theorem 3.9, we can conclude that for all $p \in X - A$, there exists an open set U about p such that $U \cap A = \emptyset \implies p \in U \subset X - A$. Since this holds for all $p \in X - A$, by Theorem 3.3 this means that $X - A$ is an open set, which is what we set out to show.

Now we prove the other direction, and suppose that $X - A$ is an open set. Since $X - A$ is open, we know that for every point $q \in X - A$, there exists an open set U of q such that $U \subset X - A$ and therefore $U \cap A = \emptyset$. Thus we see that none of the $q \in X - A$ could possibly be a limit point of A since every point of $X - A$ violates the definition of a limit point of A. Thus all the limit points of A must be in A, so that A is closed. With both directions proven, the theorem is itself proved.

$\blacksquare$

Theorem 3.15 Let $(X, \mathcal{T})$ be a topological space, and let U be an open set and A be a closed subset of X. Then the set $U - A$ is open and $A - U$ is closed.

Proof: We can show that $U - A$ is open as follows. Since A is closed, we know that $X - A$ must be an open set by Theorem 3.14. Now $U - A = U \cap (X - A)$, so $U - A$ is the intersection of two open sets and hence is itself an open set, which is what we set out to show.

Next, observe that $A - U = A \cap (X - U)$. Thus $A - U$ is the intersection of two closed sets, which implies that $A - U$ is itself closed, as desired.

$\blacksquare$

Theorem 3.16 Let $(X, \mathcal{T})$ be a topological space. Then:
i) $\emptyset$ is closed.
ii) X is closed.
iii) The union of finitely many closed sets is closed.
iv) Let $\{A_\alpha\}_{\alpha \in \lambda}$ be a collection of closed subsets in $(X, \mathcal{T})$. Then $\cap_{\alpha \in \lambda} A_\alpha$ is closed.

Proof: We can first prove (i) by observing that, since the empty set contains no elements, it is vacuousely true that it contains all of its limit points. Thus $\emptyset$ is a closed set.

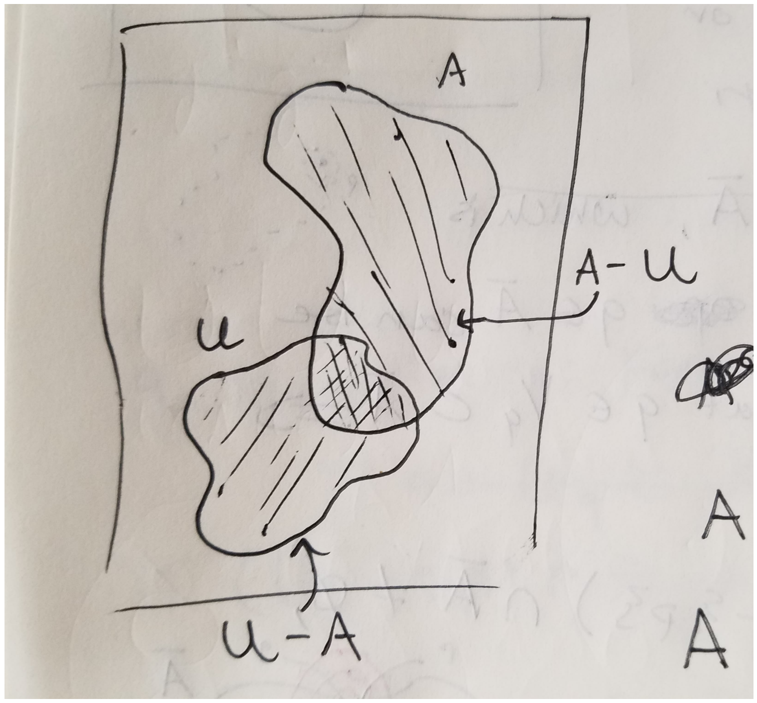


Figure 1: Two arbitrary sets U and A are drawn as well as the sets $A - U$ and $U - A$.

To prove (ii), observe that X, the entire space, must contain all of its limit points. Thus X is a closed set.

For (iii), let p be a limit point of $\bigcup_{i=1}^{n} A_i$. Then for at least for every neighborhood U of p we have that $(U - \{p\}) \cap \bigcup_{i=1}^{n} A_i \neq \emptyset$ so that $(U - \{p\}) \cap A_i \neq \emptyset$ for at least one A_i in of $\{A_i\}_{i=1}^{n}$. Thus all the limit points of $\bigcup_{i=1}^{n} A_i$ are simply limit points of the sets in $\{A_i\}_{i=1}^{n}$. Thus $\bigcup_{i=1}^{n} A_i$ contains all of its limit points so it is a closed set.

demorgans laws

To prove (iv), consider an arbitrary collection of closed sets $\{A_\alpha\}_{\alpha\in\lambda}$, where λ is an arbitrary index. Observe that by DeMorgan's Laws

$$\left(\bigcap_{\alpha\in\lambda} A_\alpha\right)^c = \bigcup_{\alpha\in\lambda} A_\alpha^c.$$

Observe that each A_α^c is an open set by Theorem 3.14, and because the arbitrary union of open sets is open, we can then conclude that $\bigcup_{\alpha\in\lambda} A_\alpha^c$ is an open set. Since $\left(\bigcap_{\alpha\in\lambda} A_\alpha\right)^c =$

$\bigcup\limits_{\alpha \in \lambda} A_\alpha^c$, Theorem 3.14 tell us that $\bigcap\limits_{\alpha \in \lambda} A_\alpha$ is a closed set, as desired.

■

Exercise 3.17 Give an example to show that the union of infinitely many closed sets in a topological space may be a set that is not closed.

Solution: On the standard topology of $\mathbb{R}$, we can take the example that $\cup_{n=1}^{\infty}[-n, n]$. The resulting set is no longer a closed set, since for every point in the resulting set we can construct a neighborhood about every point such that the neighborhood is entirely contained in the set.

□

Exercise 3.18 Give examples of topological spaces and sets in them that:
1. are closed, but not open;
2. are open, but not closed;
3. are both open and closed;
4. are neither open nor closed.

Solution:

1. In the standard topology on $\mathbb{R}$, closed sets are definitely not open sets.

2. Again, in the standard topology, open sets are not the same thing as closed sets. We can also use the example of the discrete topology, since every subset is considered to be an open set. None of the sets are closed.

3. In the indiscrete topology, every set is both open and closed since each set simultaneously contains all of its limit points and every point in each set can be contained in a ball which is a subset of the respective set.

4. Consider the finite complement topology on $\mathbb{R}$. The set $\mathbb{Z}$ is not open or closed in this topology.

□

Exercise 3.19 State whether each of the following sets are open, closed, both, or niether.
1. In $\mathbb{Z}$ with the finite complement topology: $\{0, 1, 2\}, \{\text{prime numbers}\}, \{n : |n| \geq 10\}$
2. In $\mathbb{R}$ with the standard topology: (0,1), (0,1], [0,1], $\{0, 1\}$, $\{\frac{1}{n} : n \in \mathbb{N}\}$.
3. In $\mathbb{R}^2$ with the standard topology: $\{(x, y) : x^2 + y^2 = 1\}, \{(x, y) : x^2 + y^2 > 1\}, \{(x, y) : x^2 + y^2 \geq 1\}$.

Solution:

1. The set $\{0, 1, 2\}$ is not an open set. Furthermore, it cannot be a closed set since it has no limit points (or does this vacuousely prove that it is a closed set?). The prime numbers are also not an open set. The set $\{n : |n| \geq 10\}$ is definitely an open set since $\mathbb{Z} - \{n : |n| \geq 10\} = \{-9, -8, \ldots, 8, 9\}$

2. (0,1) is an open set in this topology since every point $x \in (0, 1)$ can be contained in a neighborhood which is a subset of $(0, 1)$.

   (0, 1] is neither an open or closed since, since it doesn't contain all of its limit points and not every point can be in a neighborhood entirely contained in the set.

   $[0, 1]$ is a closed set since it contains all of its limit points.
   $\{0, 1\}$ is not an open set because not every neighborhood containing either 0 or 1 will be entirely contained in the set. It is also not a closed set since it doesn't have any limit points (or does this imply that it can be a closed set?).

   Finally, $\{\frac{1}{n} : n \in \mathbb{N}\}$ is not a closed set because it doesn't contain its one limit point, 0. It is also not an open set because not every neighborhood of every point of the set can be entirely contained in the set.

3. The set $\{(x, y) : x^2 + y^2 = 1\}$ cannot be open since not every open set about an element of the set will be entirely contained in the set. It is however open because it contains all of its limit points.

   The set $\{(x, y) : x^2 + y^2 > 1\}$ is open because every point can be contained by an open set which is in turn contained in the entire set. It is not closed because it does not contain its limit points.

Finally, the set $\{(x, y) : x^2 + y^2 \geq 1\}$ is closed because it contains all of its limit points which lie on the circle.

$\square$

Theorem 3.20 For any set A in a topological space X, the closure of A equals the intersection of all closed sets containing A, that is,

$$\overline{A} = \bigcap_{A\subset B, B\in \mathcal{C}} B$$

where $\mathcal{C}$ is the collection of all closed sets in X.

Proof: Observe that $\overline{A}$ is a closed set which contains A so that $\overline{A} \in \mathcal{B}$. Thus we'll have that $\bigcap\limits_{A\subset B, B\in \mathcal{C}} B \subset \overline{A}$. Next observe that for all $B \in \mathcal{C}$, $\overline{A} \subset B$. This is because $\overline{A}$ is the smallest closed set which contains A. We can argue this by noting that if we delete any point from $\overline{A}$, we'd either delete a point of A and we'd no longer contain A, or we'd delete a limit point of A and our set would no longer be closed. Hence $\overline{A}$ is the smallest closed set containing A.

Since $\overline{A} \subset B$ for all $B \in \mathcal{C}$, we can then state that $\overline{A} \subset \bigcap\limits_{A\subset B, B\in \mathcal{C}} B$. Since we already showed that $\cap_{A\subset B, B\in \mathcal{C}} B \subset \overline{A}$, this becomes sufficient to prove that $\overline{A} = \bigcap\limits_{A\subset B, B\in \mathcal{C}} B$.

$\blacksquare$

Exercise 3.21 Pick several different subsets of $\mathbb{R}$ and find their closures in:

1. the discrete topology;
2. the indiscrete topology;
3. the finite complement topology;
4. the standard topology.

Solution:

---

1. Consider the $(0,1)$. Then in the discrete topology, we know that the closure is just the set itself, because every set in the discrete topology is closed.

2. In the indiscrete topology. the closure of the set is all of $\mathbb{R}$, since every point of $\mathbb{R}$ is a limit point of $(0,1)$.

3. In this case, every point of $\mathbb{R}$ is also a limit point to the finite complement topology, since every open set will always contain points in $(0,1)$ because it is uncountably infinite.

4. In the standard topology, $[0,1]$ would be the closure of the set since $0,1$ are the limit points of the set.

$\square$

Theorem 3.22. Let A and B be subsets of a topological space X. Then

1. $A \subset B$ implies $\overline{A} \subset \overline{B}$
2. $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

Proof: Consider a limit point p of A. By definition, for every open set U of p, we have that $(U - \{p\}) \cap A \neq \emptyset$. However, since B contains A, we can also state that $(U - \{p\}) \cap B \neq \emptyset$, meaining that p must also be a limit point of B. Thus $\overline{A} \subset \overline{B}$.

Consider limit points p, q of A, B respectively. Then for all open sets U, V containing p, q respectively, we'll have that $(U - \{p\}) \cap A \neq \emptyset$ and $(V - \{q\}) \cap B \neq \emptyset$. Now it is definitely true that for these same open sets that $(U - \{p\}) \cap (A \cup B) \neq \emptyset$ and $(V - \{q\}) \cap (A \cup B)$, so that both p and q must be limit points of $A \cup B$. Thus $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$.

Now suppose that r is a limit point of $\overline{A \cup B}$. Then this means that for every open set W of r, we have that $(W - \{r\}) \cap (A \cup B) \neq \emptyset$. Thus either $(W - \{r\}) \cap A \neq \emptyset$, or $(W - \{r\}) \cap B \neq \emptyset$ or both. In other words, r is either a limit point of A, B, or both. In any case, this implies that $r \in \overline{A} \cup \overline{B}$, so that what we have is that $\overline{A \cup B} \subset \overline{A} \cup \overline{B}$. Since we already showed that $\overline{A} \cup \overline{B} \subset \overline{A \cup B}$, this effectively proves that $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

$\blacksquare$

Exercise 3.23 Let $\{A_\alpha\}_{\alpha\in\lambda}$ be a collection of subsets of a topological space X. Then is the following statement true?

$$\overline{\bigcup_{\alpha\in\lambda} A_\alpha} = \bigcup_{\alpha\in\lambda} \overline{A}_\alpha$$

Solution: The statement is false. Consider the sequence of sets $A_n = \{[\frac{1}{n}, 1] : n \in \mathbb{N}\}$. While

$$\bigcup_{n=1}^{\infty}\left[\frac{1}{n}, 1\right] = (0,1] \implies \overline{\bigcup_{n=1}^{\infty}\left[\frac{1}{n}, 1\right]} = [0,1]$$

We see that

$$\bigcup_{n=1}^{\infty}\overline{\left[\frac{1}{n}, 1\right]} = \bigcup_{n=1}^{\infty}\left[\frac{1}{n}, 1\right] = (0,1].$$

Thus this is a counterexample since obviously $(0,1] \neq [0,1]$.

$\square$

Exercise 3.24 In $\mathbb{R}^2$ with the standard topology, describe the limit points and closure of each of the following two sets:

1. $S = \{(x, \sin(\frac{1}{x})) : x \in (0,1)\}$
2. $C = \{(x,0) : x \in [0,1]\} \cup \bigcup\limits_{n=1}^{\infty}\{(\frac{1}{n}, y) : y \in [0,1]\}$

Solution: Note: The topologist sine curve can be connected or not connected, depending on what definition you're using.

For (1), we can graph the function to see that there is rapid oscillations as x approaches the origin. The function rapidly changes from -1 to 1, and does so indefinitely as x approaches 0. Thus we can say that $\{(0,y) : y \in [-1,1]\}$ is the set of limit points, so

$$\left\{\left(x, \sin\left(\frac{1}{x}\right)\right) : x \in (0,1)\right\} \cup \{(0,y) : y \in [-1,1]\}$$

is the closure of the set.

The topologist comb is connected in both definitions of connectivity.

For the comb, we can see that a series of lines converge to the interval $\{(0,y) : y \in [0,1]\}$ as x approaches 0 from the right, this must be the set of limit points. Thus the closure must be

$$\{(x,0) : x \in [0,1]\} \cup \bigcup_{n=1}^{\infty}\left\{\left(\frac{1}{n}, y\right) : y \in [0,1]\right\} \cup \{(0,y) : y \in [0,1]\}.$$

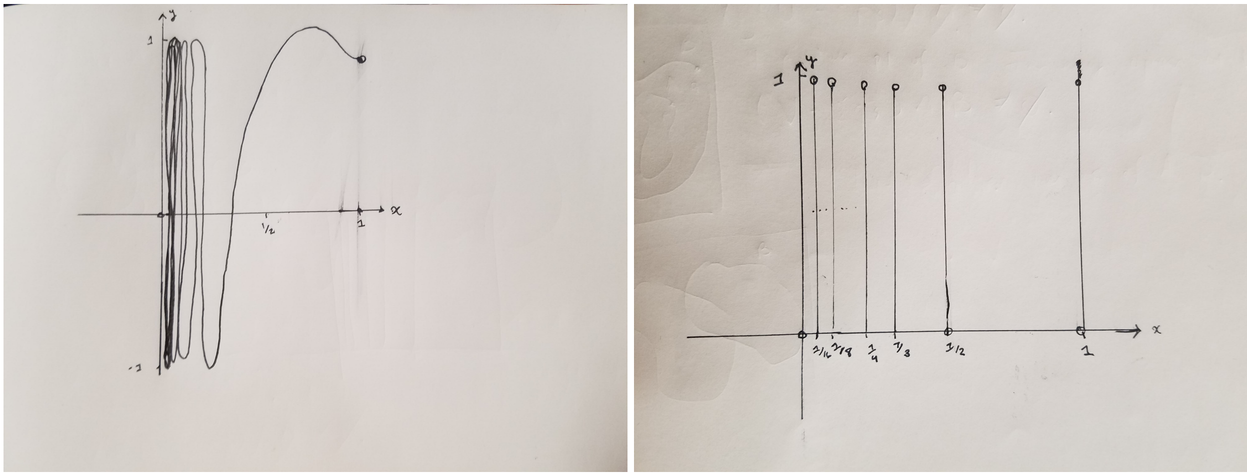


Figure 2: The lefthand drawing is the topologist's sine curve, while the right hand drawing is the topologist's comb.

$\square$

Exercise 3.25 In the standard topology on $\mathbb{R}$, there exists a non-empty open subset C of the closed unit interval $[0, 1]$ that is closed, contains no non-empty open interval, and where no point of C is an isolated point.

Solution: The rationals won't work because rationals aren't closed, since their limit points are irrationals.

Consider the Cantor set. Everytime you try to construct an open interval it will eventually be able to escape and no longer be contained in the cantor set. It is closed because it contains an arbitrary intersection of closed sets.

$\square$

Theorem 3.26 Let A be a subset of a topological space X. Then p is an interior point of A if and only if there exists an open set U with $p \in U \subset A$.

Proof: First we start with the forward direction. Suppose that for some $p \in A$, there exsits an open set U such that $p \in U \subset A$. Since U is open and $U \subset A$, by definition we have that $U \subset \text{Int}(A)$, and therefore $p \in \text{Int}(A)$. Thus p must be an interior poiint.

Now suppose that p is an interior point of A. Then since $p \in \text{Int}(A) = \bigcup\limits_{U \subset A, U \in \mathfrak{T}} U$, we know that for at least one $U \in \mathfrak{T}$, $p \in U \subset A$. Thus there exists an open set U containing p which is a subset of A, which is what we set out to show. With both directions proved, we have proved the theorem.

■

Exercise 3.27 Show that a set U is open in a topological space X if and only if every point of U is an interior point of U.

Solution: We'll first prove the forward direciton. Let $U \subset X$, and suppose every point in U is an interior point of U. By Theorem 3.26 for all $p \in U$ there exists an open set V_p such that $p \in V_p \subset U$. Since every point $p \in U$ is contained in an open ball V_p which is a subset of U, we have that U must be an open set by Theorem 3.3.

Now we prove the other direction. Suppose that U is an open set. Then by Theorem 3.3, for every point $p \in U$, there exists an open ball V_p such that $p \in V_p \subset U$. But by Theorem 3.26, this means that every $p \in U$ is an interior point of U, which is what we set out to show.

□

Theorem 3.28 Let A be a subset of a topological space X. Then $\text{Int}(A)$, $\text{Bd}(A)$ and $\text{Int}(X - A)$ are disjoint sets whose union equals X.

Proof: First we'll show that these sets are disjoint. Consider a point $p \in \text{Int}(A)$. By theorem 3.26, there exists an open ball U of p such that $p \in U \subset A$. Therefore, $p \notin \overline{X - A}$. This is because $p \notin (X - A)$, and p is not a limit point of this set because not every open set of p intersects with $X - A$. Namely, the open set $U \subset A$ which we constructed earlier contains p but does not intersect $X - A$. Therefore $p \notin \overline{X - A}$.

This fact helps us in two ways. Since $p \notin \overline{X - A}$, it is definitely true that $p \notin \text{Int}(X - A) \subset \overline{X - A}$, and that $p \notin \text{Bd}(A)$ since the definition of $\text{Bd}(A)$ is $\overline{A} \cap \overline{X - A}$. Thus $\text{Int}(A)$ is disjoint with $\mathrm{Bd}(A)$ and $\mathrm{Int}(X-A)$.

Finally we'll show that $\mathrm{Int}(X-A)$ is disjoint with $\mathrm{Bd}(A)$. Let $q \in \mathrm{Int}(X-A)$. By Theorem 3.26 there exists an open set U_q such that $q \in U_q \subset (X-A)$. Thus q cannot be in $\overline{A}$. We can then conclude that $q \notin \mathrm{Bd}(A)$ because $\mathrm{Bd}(A) = \overline{A} \cap \overline{X-A}$, and we just showed that $q \notin \overline{A}$. Therefore, $\mathrm{Int}(X-A)$ is disjoint with $\mathrm{Bd}(A)$.

Now for the sake of contradiciton, suppose there exists a point $r \in X$ such that $r \notin \mathrm{Int}(A) \cup \mathrm{Bd}(A) \cup \mathrm{Int}(X-A)$. Since $r \notin \mathrm{Int}(A)$ and $r \notin \mathrm{Int}(X-A)$, then by definition, we know that every open set containing r must intersect A and $X-A$. But this would imply that $r \in \mathrm{Bd}(A)$, which is a contradiction. Thus there is no $r \in X$ such that $r \notin \mathrm{Int}(A) \cup \mathrm{Bd}(A) \cup \mathrm{Int}(X-A)$, which means that $X = \mathrm{Int}(A) \cup \mathrm{Bd}(A) \cup \mathrm{Int}(X-A)$.

■

Exercise 3.29

Exercise 3.29 Pick several different subsets of $\mathbb{R}$, and for each one, finds its interior and boundary using:

1. the discrete topology;
2. the indiscrete topology;
3. the finite complement topology;
4. the standard topology.

Solution:

1. Consider the set $(0,1)$. Since this is the discrete topology, we know that every subset of $\mathbb{R}$ is open. Therefore, the interior of $(0,1)$ is simply itself. The boundary of this set is simply empty, since $\overline{(0,1)} \cap \overline{R-(0,1)} = \emptyset$.

2. For $(0,1)$, the interior is $\emptyset$, since the empty set is the largest set contained in $(0,1)$. On the other hand, the boundary is simply the set $\{0,1\}$ since $(0,1) \cap \mathbb{R} - (0,1) = \{0,1\}$.

3. On the finie complement topology, $(0,1)$ does not have an interior. This is because on this topology there does not exist an open set contained in $(0,1)$. The set is also not closed, since it does not contain its limit points. In fact, every point in $\mathbb{R}$ is a limit point of the set, so $\overline{(0,1)} = \mathbb{R}$ and $\overline{\mathbb{R} - (0,1)} = \mathbb{R}$, so the boundary is simply $\mathbb{R}$.

4. For the standard topology, the interior is simply $(0,1)$. The boundary is $\{0,1\}$, since $\overline{(0,1)} \cap \overline{\mathbb{R} - (0,1)} = \{0,1\}$.

□

Theorem 3.30 Let A be a subset of the topological space X and let p be a point in X. If $\{x_i\}_{i \in \mathbb{N}} \subset A$ and $x_i \to p$, then p is in the closure of A.

Proof: Since $x_i \to p$, we know that for every open set U containing p, there exists an $N \in \mathbb{N}$ such that $x_i \in U$ for $i > N$. However, for all $n \in \mathbb{N}$, we know that $x_n \in A$. Therefore, we know that $(U - \{p\}) \cap A \neq \emptyset$ for any open set U containing p. Thus p must be a limit point of A, so p is in the closure of A.

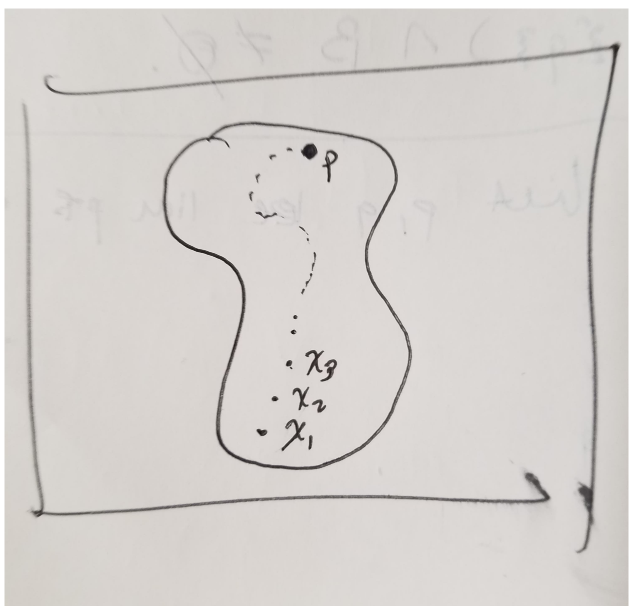


Figure 3: With the drawing, its easy to see that if all the points in the sequence must be in A, then the limit of the sequence is at most in the closure of A.

■

Theorem 3.31 In the standard topology on $\mathbb{R}^n$, if p is a limit point of a set A, then there is a sequence of points in A that converge to p.

Proof: Since p is a limit point of A, we know that for every open set U containing p, we have that $(U - \{p\}) \cap A \neq \emptyset$. Thus let $\epsilon > 0$, and consider the sequence of balls $B(p, \epsilon/n)$ containing p of radius ϵ/n. Then since $(B(p, \epsilon/n) - \{p\}) \cap U \neq \emptyset$, for each n there must exist a $q \in A$ such that $q \in (B(p, \epsilon/n) - \{p\})$. Label these such q as q_n.

Now let $\delta > 0$, and consider the open ball $B(p, \delta)$. Then there exists an $m \in \mathbb{N}$ such that $\epsilon/m < \delta$ so that $B(p, \epsilon/m) \subset B(p, \delta)$. In other words, for any open set U about p, there exists a number $m \in \mathbb{N}$ such that for all $n > m$, $q_n \in U$. Therefore we can conclude that $\{q_n\}$ is a sequence of points where for all $n \in \mathbb{N}$, $q_n \in A$ and $q_n \to p$, which is what we set out to show.

In general, the limit point of a set is not the same thing as the limit of a sequence.

■

Exercise 3.32 Find an example of a topological space and a convergent sequence in that space, where the limit of the sequence is not unique.

Solution: An easy example can be found with the indiscrete topology on $\mathbb{R}$. Consider the sequence $1, 2, 3, \ldots$. Then every $x \in \mathbb{R}$ is a limit of the sequence, since the only open set containing any point is $\mathbb{R}$ which obviously contains every point of the sequence.

□

Exercise 3.33 1. Consider sequences in $\mathbb{R}$ with the finite complement topology. Which sequences converge? To what value(s) do they converge?
2. Consider sequences in $\mathbb{R}$ with the countable complement topology. Which sequences converge? To what value(s) do they converge?

Solution: Consider the sequence $\left\{\frac{1}{n} | n \in \mathbb{R}\right\}$ and $\left\{\frac{n}{n+1} | n \in \mathbb{N}\right\}$. Then on the finite complement topology, we see that both sequences are convergent. This is because, for either of the sequences, we cannot construct an open set around a limit point which does not include points of the sequence, since every open set is of the form $\mathbb{R} - X$, where X is a finite set, and both sequences are countably infinite. In addition, the convergence of both sets is not unique, since on this topology, every open set of any $x \in \mathbb{R}$ will inevitably include points of the sequences.

In both cases, neither of the sequences are convergent. This is because for any $x \in \mathbb{R}$ which could be a limit point of the sequence, we can construct an open set U containing x where $U = \mathbb{R} - \left\{\frac{1}{n} | n \in \mathbb{R}\right\}$. Thus by the definition of a limit of a sequence, neither of these points converge.
Note: what is the relationship between the sequences which are and aren't convergent on these two different topological spaces? It seems like a finite sequence wouldn't be convergent on the finite complement topology, while a infinite one is, and that an infinite sequence isn't convergent on the countable complement topology, while a finite one is.

$\square$

Chapter 4

Bases, Subspaces, Prodcuts: Creating New Spaces

Theorem 4.1 Let $(X, \mathcal{T})$ be a topological space and $\mathcal{B}$ be a collection of subsets of X. Then $\mathcal{B}$ is a basis for $\mathcal{T}$ if and only if:
1. $\mathcal{B} \subset \mathcal{T}$
2. for each set U in $\mathcal{T}$ and point p in U there is a set V in $\mathcal{B}$ such that $p \in V \subset U$

Proof: First we prove the forward direction. Suppose we have a set $\mathcal{B}$ such that $\mathcal{B} \subset \mathcal{T}$, and for every open set $U \in \mathcal{T}$ and point p in U there is a set V in $\mathcal{B}$ such that $p \in V \subset U$. Then let A be an open set of X. For all $a \in A$, there exists an open set $V_a \in \mathcal{B}$ such that $a \in V_a \subset A$. Then observe that

$$\bigcup_{a \in A} V_a = A.$$

Thus we see that every open set in X is the union of elements of $\mathcal{B}$, so $\mathcal{B}$ is a basis for X.

Now we prove the other direction, and suppose $\mathcal{B}$ is a basis for X. First observe that $\mathcal{B} \subset \mathcal{T}$ because this is part of the definition of a basis, so this proves (1). Now let $U \in \mathcal{T}$. Then

$$\bigcup_{B \in \mathcal{B}'} B = U$$

for some subset $\mathcal{B}'$ of $\mathcal{B}$. Thus for any $p \in U$, there must exist at least one $B \in \mathcal{B}'$ such that $p \in B$ and by construction $B \subset U$. Therefore, we have that for each set U in $\mathcal{T}$ and point p in U there is a set B in $\mathcal{B}$ such that $p \in B \subset U$, as desired.

■

Ex: the set of length 1 intervals. This does not generate a topology on $\mathbb{R}$ because intersections should be open but sometimes they are intervals of length less than 1. Another topology is the stick bubble topology. Open sets are balls sitting in the upper half plane or an open ball containing a single point on its circumference which is shared with the boundary of the upper half plane.

Ordered set topology:

Exercise 4.2
1. Let $\mathcal{B}_1 = \{(a,b) \subset \mathbb{R} : a, b \in \mathbb{Q}\}$. Show that $\mathcal{B}_1$ is a basis for the standard topology on $\mathbb{R}$.
2. Let $\mathcal{B}_2 = \{(a,b) \cup (c,d) \subset \mathbb{R} : a < b < c < d$ are distinct irrational numbers.$\}$ Show that $\mathcal{B}_2$ is also a basis for the standard topology on $\mathbb{R}$.

Solution: 1. We can do this using Theorem 4.1. Observe firstly that $\mathcal{B}_1 \subset \mathcal{T}_{\text{std}}$. Now consider an arbitrary open set U in $\mathbb{R}$. By definition, for any $p \in U$ there exists an open ball $B(p, \epsilon(p))$ such that $p \in B(p, \epsilon(p)) \subset U$. Since the rationals are dense in $\mathbb{R}$, for any $p \in U$ there must exist a rationals a/b and c/d such that

$$p - \epsilon(p) < a/b < p < c/d < p + \epsilon(p).$$

Thus $(a/b, c/d) \in \mathcal{B}_1$ and $p \in (a/b, c/d) \subset B(p, \epsilon) \subset U$. This proves (2) of the theorem, so $\mathcal{B}_1$ is a basis for the standard topology on $\mathbb{R}$.

2. We can do this using Theorem 4.1. Observe firstly that $\mathcal{B}_1 \subset \mathcal{T}_{\text{std}}$. Now consider an arbitrary open set U in $\mathbb{R}$. By definition, for any $p \in U$ there exists an open ball $B(p, \epsilon(p))$ such that $p \in B(p, \epsilon(p)) \subset U$. Since the irrationals are dense in $\mathbb{R}$, for any $p \in U$ there must exist a rationals a, b, c and d such that

$$p - \epsilon(p) < a < p < b < c < d < p + \epsilon(p).$$

Thus $(a,b) \cup (c,d) \in \mathcal{B}_2$ and $p \in (a,b) \cup (c,d) \subset B(p, \epsilon) \subset U$. This proves (2) of the theorem, so $\mathcal{B}_2$ is a basis for the standard topology on $\mathbb{R}$.

$\square$

Theorem 4.3 Suppose X is a set and $\mathcal{B}$ is a collection of subsets of X. Then $\mathcal{B}$ is a basis for some topology on X if and only if:
1. Each point of X is in some element $\mathcal{B}$
2. if U and V are sets in $\mathcal{B}$ and p is a point in $U \cap V$, there is a set W of $\mathcal{B}$ such that $p \in W \subset (U \cap V)$.

Proof: First we'll prove the forward direction. Suppose that $\mathcal{B}$ is a basis for some topology $\mathcal{T}$. Since $X \in \mathcal{T}$, we know by Theorem 4.1 that for every $p \in X$ there exists a $B \in \mathcal{B}$ such that

$$p \in \mathcal{B} \subset X.$$

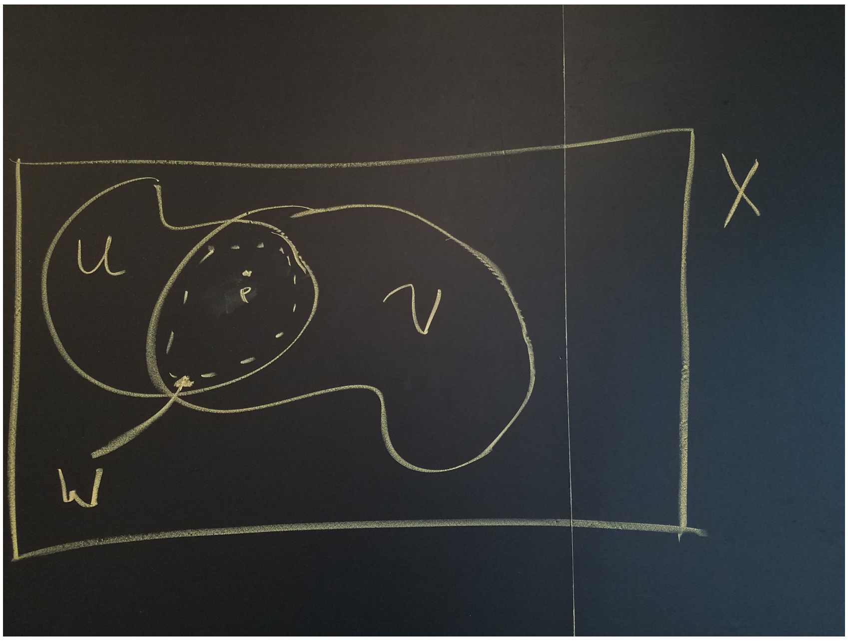


Figure 1: Here we have sketched the conditions (1) and (2), where X is the whole space, $U, V \in \mathcal{B}$ and $W \in (U \cap V)$.

Thus this proves (1). Next observe that if $U, V \in \mathcal{B}$, then $U, V \in \mathcal{T}$. Therefore, $U \cap V \in \mathcal{T}$ is an open set in X. By Theorem 4.1, for any $p \in U \cap V$, there must exist a $W \in \mathcal{B}$ sucht that

$$p \in W \subset U \cap V.$$

This proves (2) which finishes the proof in this direction.

Suppose that (1) and (2) are true. Then consider the set of all possible unions of elements of $\mathcal{B} = \{B_\alpha\}_{\alpha \in \lambda}$, namely the set

$$\mathcal{T} = \{\bigcup_{\alpha \in \lambda'} B_\alpha : \lambda' \subset \lambda\}.$$

We can now verify the properties that this is a topology.

1. Observe that $\emptyset \in \mathcal{T}$ if we take an empty union of objects.

2. $X \in \mathcal{T}$ by condition (1).

3. Observe that arbitrary unions of elements of $\mathcal{B}$ are within $\mathcal{T}$, since that is by definition how we constructed $\mathcal{T}$.

4. By condition (2) if $U, V \in \mathcal{B}$ then there exsits a W such that $W \in \mathcal{B}$ and $p \in W \subset (U \cap V)$. Thus observe that

$$U \cap V = \bigcup_{p \in U \cap V} W_p$$

where $W_p \in \mathcal{B}$ such that $p \in W_p \subset U \cap V$. By our definition of our topology, $U \cap V$ must be an open set. Therefore, finite interesections of open sets are open.

Thus we have shown that conditions (1) and (2) generate a topology, which completes the proof.

■

Exercise 4.4 Show that the basis proposed above (all sets of the form $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$) for the lower limit topology is in fact a basis.

Solution: We can show that this is a basis by using Theorem 4.3. Observe that for any point $p \in \mathbb{R}$, there exists $a, b \in \mathbb{R}$ such that $a \leq p < b$. Thus every point in $\mathbb{R}$ is in some element of our basis. Next, let $c < d < e < f$ and consider $U = [c, d]$ and $V = [e, f]$. Then $U \cap V = \emptyset \in \mathbb{R}_{\text{LL}}$.
Next, let $c < d$ and $e < d < f$, and consider the sets $U = [c, d)$ and $V = [e, f)$. Then there is a point $p \in (U \cap V)$. Observe that if $\epsilon < p - e$ then the set $p \in [p - \epsilon, f) \subset U \cap V$ and $[p - \epsilon, f)$ is a member of our basis. Thus we have that sets of the form $[a, b)$ form a basis for the lower limit topology.

□

Theorem 4.5 Every open set in $\mathbb{R}_{\text{std}}$ is an open set in $\mathbb{R}_{\text{LL}}$, but not vice versa.

Proof: Consider an open set $B(p, \epsilon)$ in $\mathbb{R}_{\text{std}}$ about a point p of radius $\epsilon > 0$, which is really just an interval $(p - \epsilon, p + \epsilon)$. Then consider the sequence of open sets $\mathbb{R}_{\text{LL}}$:

$$\left\{ \left[p - \epsilon \left(1 - \frac{1}{2^n} \right), p + \epsilon \right) : n \in \mathbb{N} \right\}.$$

Recall that an arbitrary union of open sets is open. Then

$$\bigcup_{n \in \mathbb{N}} \left[p - \epsilon \left(1 - \frac{1}{2^n} \right), p + \epsilon \right) = (p - \epsilon, p + \epsilon)$$

is an open set. Thus we have that open sets in $\mathbb{R}_{\text{std}}$ are open in $\mathbb{R}_{\text{LL}}$. However, it is obvious that open sets in $\mathbb{R}_{\text{LL}}$ are not open in $\mathbb{R}_{\text{std}}$, because sets of the form $[a, b)$ are neither open or closed in $\mathbb{R}_{\text{std}}$. Thus this proves the theorem.

■

Exercise 4.6 Give an example of two topologies on $\mathbb{R}$ such that neither is finer than the other, that is, the two topologies are not comparable.

Solution: We can define an upper limit topology $\mathbb{R}_{\text{UL}}$ generated by basis sets $(a, b] = \{x \in \mathbb{R} | a < x \leq b\}$.

For each $x \in \mathbb{R} \in (x - \epsilon, x + \epsilon] \in \mathbb{R}_{\text{UL}}$. In addition, observe that $(a, b] \cap (c, d] = (b, c]$ if $b < c$ and $(a, d]$ if $b = c$ and $\emptyset$ if $b > c$, all of which are basic open sets in the topology. Thus by Theorem 4.3 this generates a topology.

Now observe that neither of the topologies $\mathbb{R}_{\text{LL}}$ and $\mathbb{R}_{\text{UL}}$ are finer than the other, since neither is a subset of the other. Thus these topologies on $\mathbb{R}$ are not comparable.

$\square$

Exercise 4.7 Check that the collection of sets that we specify as a basis in the double headed snake actually forms a basis for the topology.

Solution: We can verify this using theorem 4.3. Observe first that every point in $\mathbb{R}_{+00}$ is contained within some set in the basis. Next, let U and V be any two sets in the topology. Then if U, V are of the form $(0, b) \cup \{0'\}$, then their intersection will be of the form $(0, a) \cup \{0'\}$ where $a \leq b$ which is a set within our basis. The argument applies again to if U, V are both of the form $(0, b) \cup \{0''\}$ or (a, b).

Next observe that if we intersect U of the form $(0, b) \cup \{0'\}$ with V of the form (a, b) then the intersection is either empty, or of the form (a, b), which is a type of set contained in our absis. If we intersct
Let U be any set in $\mathbb{R}_{+00}$. Let $U = (a, b)$ where $a, b \in \mathbb{R}$ and $a < b$. If U does not contain $\{0'\}$ or $\{0''\}$, then there exits numbers $c, d \in mathbbR$ such that $a < c < d < b$. Then the set (c, d) is in our basis and is a subset of U.

$\square$

Exercise 4.8 In the Double Headed Snake, show that every point is a closed set; however, it is impossible to find disjoint open sets U and V such that $\{0'\} \in U$ and $\{0''\} \in V$.

Solution: Observe that the complement of every point is an interval of the line, for which we can represent as the union of basic open sets and therefore the complement of every point is open. Thus every point must be a closed set.

Next, let U be an open set containing $\{0'\}$ and V an open set containing $\{0''\}$. Then by definition, there must exist basic open sets U_B, V_B such that $0' \in U_B \subset U$ and $0'' \in V_B \subset V$. Since they are basic open sets containing the zeros of the double headed snake, both are either of the form $(0,b) \cup \{0'\}$ or $(0,b) \cup \{0''\}$, so that $U_B \cap V_B \neq \emptyset$. Thus we cannot find disjoint open sets U and V such that $\{0'\} \in U$ and $\{0''\} \in V$.

□

Exercise 4.9 1. In the topological space $\mathbb{R}_{\text{har}}$, what is the closure of the set $H = \{1/n\}_{n\in\mathbb{N}}$?
2. In the topological space $\mathbb{R}_{\text{har}}$, what is the closure of the sets $H^- = \{-1/n\}_{n\in\mathbb{N}}$?
3. Is it possible to find disjoint open sets U, V in $\mathbb{R}_{\text{har}}$ such that $0 \in U$ and $H \subset V$?

Solution:

1. There are no limit points to the set because all sets are of the form (a,b) or $(a,b) - H$. Thus for any neighborhood about a point will always either not contain H or it will exclude H, so by definition no point can be a limit point of H.

2. For H^-, the limit points just consists of the set $\{0\}$. This is any open set which contains 0 must contain points of the sequence $\{\frac{1}{n} | n \in \mathbb{N}\}$. The difference between this question and question (1.) is that in the first question, the points in the sequence were always excluded whenever they interesected any open set containing 0, whereas that's not the case here since we don't care about excluding the negative harmonics.

3. For any open set U containing 0, we must have that there exists an open set U_0 of 0 such that $0 \in V \subset U$. And since the basic open sets are (a,b) or $(a,b) - H$, there must exist points to the right of 0 in the set U; otherwise, 0 would be a limit point. Because any open set containing H must contain points arbitrarily close to 0, it is inevitable for U and V to intersect. This is because for any open set which contains 0, say $(a, 0+\epsilon)$ for $\epsilon > 0$ and $a < 0$, there exsits an $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$ and therefore any open set containing H must contain this point and hence intersect with $(a, 0+\epsilon)$. Of course, any set of the form $(0-\epsilon, b)$ for $b > 0$ will intersect with

any set contanining H by the same argument as before. Thus there does not exist disjoint open sets U, V such that $0 \in U$ and $H \subset V$.

$\square$

Exercise 4.10 1. In $\mathbb{H}_{\text{bub}}$, what is the closure of the set of rational on the x-axis?

2. In $\mathbb{H}_{\text{bub}}$, which subsets of the x-axis are closed?

3. In $\mathbb{H}_{\text{bub}}$, let A be a countable set on the x-axis and z a point on the x-axis not in A. Then there exist open sets U and V such that $A \subset U$ and $z \in V$. (Do you need the countability hypothesis on A?)

4. In $\mathbb{H}_{\text{bub}}$, let A and B be countable sets on the x-axis such that A and B are disjoint. Then there exists open sets U and V such that $A \subset U$ and $B \subset V$.

5. In $\mathbb{H}_{\text{bub}}$, let A be the rational numbers and let B be the irrational numbers. Do there exist disjoint open sets U and V such that $A \subset U$ and $B \subset V$?

Solution:

1. Suppose that a limit point was no a rational on the x-axis. Obviously such a point cannot be one which isn't on the x-axis since we could easily find an open set which contained that point but didn't intersect the x-axis. Thus our candidates for limit points for our set are reduced to irrationals on the x-axis. But observe that any open set which contains a rational doesn't necessarily contain an irrational. For example, any basic open set about a rational on the x-axis given by $(p, 0)$ does not contain an irrational. Thus the closure of the rationals is simply the rationals.

2. Consider the set $\{(x, y) \in \mathbb{R}^2 | y > 0\}$; that is, everything but the x-axis. Observe that this set is simply the uncountable union of all possible basic open sets of the form $B((x, y), r)$ where $0 < r \leq y$, so therefore this set must be open. Thererfore, its complement, the x-axis, must be closed. Furthermore, if we included in our uncountable union an arbitrary number of sticky bubbles of the form $B((x, y), r) \cup \{(x, 0\}$, which would form an open set, the complement would be a subset of the x-axis and it would be closed since the complement of open sets are closed. Thus all subsets of the x axis are open.

3. We argue that the countability hypothesis is not necessary. For any set U containing A, U must contain a set of sticky bubbles B_x of radius r_x containing each point $x \in A$. Then if we can construct a sticky bubble of radius $r' < \max\{\sqrt{(z-x)^2 + r_x^2} -$

$r_x^2\}$ about z, we see that U and V do not intersect. Thus there does exists disjoint open sets U and V such that $A \subset U$ and $z \in V$.

4. Let $a \in A, b \in B$, and arbitrarily assign sticky balls $\{B_a\}$ each of radius $r_a > 0$ for each $a \in A$. Then for each $b = (x_b, y_b)$, assign b a sticky ball B_b of radius r such that $r < r'$ for all

   $r' \in \{\sqrt{(x_a - x_b)^2 + r_a^2} - r_a^2 | r_a$ is the radius of the sticky ball of point $(x_a, y_a) \in A\}$.

   If we do this for each point $b \in B$, and union the sticky balls $\{B_b\}$, we'll obtain an open set V such that $B \subset V$. If we union all the sticky balls B_a, we'll again obtain an open set U such that $A \subset U$, which proves the exercise.

5.

$\square$

Exercise 4.11 Check that the arithmetic progressions form a basis of a topology on $\mathbb{Z}$.

Solution: We can use Theorem 4.3 for this. Let the set of arithmetic progression be $\mathcal{B}$ and let $q \in \mathbb{Z}$. Then observe that $q \in \{p \cdot n : n \in \mathbb{Z}\} \in \mathcal{B}$. Thus condition (1) of Theorem 4.3 is satisfied.

Now let $U, V \in \mathcal{B}$, and suppose $U = \{a_1 n + b_1 : n \in \mathbb{Z}\}$ and $V = \{a_2 n + b_2 : n \in \mathbb{Z}\}$. Suppose $U \cap V$ is nonempty. Then $q \in U \cap V$ for some $q \in \mathbb{Z}$. However, in order for q to be in the intersection, we must have tht a_1 and a_2 are coprime. If they are coprime, then by Bezout's theorem that there exist integers m_1, m_2 such that $m_1 n_1 + m_2 n_2 = 1$. We then know by the Chinese Remainder Theorem that $q = b_1 + (b_2 - b_1) m_1 a_1$. Therefore, we see that

$$q \in \{b_1 + (b_2 - b_1) n a_1 : n \in \mathbb{Z}\} \subset U \cap V.$$

Since this is an arithmetic progression, this set lies in $\mathcal{B}$. Therefore, we see that condition (3) of Theorem 4.3 is satisfied, so that the arithmetic progressions do form a basis for a topology on $\mathbb{Z}$.

$\square$

Theorem 4.12 There are infinitely many primes.

Proof: Let p be prime and consider the set $p\mathbb{Z}$. Observe that this is a closed set since it is the complement of the union of sets $p\mathbb{Z}+1,\ldots,p+(p-1)$ which are of the forms of basic open sets.

Now observe that nonempty open sets are always open. This is because every open set must contain a basic open set, which are by definition infinite sets.

Thus suppose that there are infinitely many primes $p_1, p_2, \ldots, p_n$. Then

$$\bigcup_{i=1}^{n} p_i\mathbb{Z}$$

is a closed set as it is the finite union of closed sets. However, note that

$$\left(\bigcup_{i=1}^{n} p_i\mathbb{Z}\right)^c = \{-1, 1\}.$$

This should be an open set, since $\bigcup_{i=1}^{n} p_i\mathbb{Z}$ is closed. But this is a contradiction since $\{-1, 1\}$ is a finite set and hence cannot be open. Thus there must be an infinite number of primes.

■

Exercise 4.18 Let X be totally ordered by $<$. Let $\mathcal{S}$ be the collection of sets of the following forms

$$\{x \in X | x < a\} \quad \text{or} \quad \{x \in X | x > a\}$$

for $a \in X$. Then $\mathcal{S}$ forms a subbasis for the order topology on X.

Solution: We can prove this using Theorem 4.14. Observe that the first condition is satisfied because $\mathcal{S} \subset \mathcal{T}$. Next, let $p \in U \in \mathcal{T}$, and suppose that U is of the form $\{x \in X | x < a\}$ or $\{x \in X | a < x\}$. Then observe that $U \in \mathcal{S}$ so that $\bigcap\limits_{n=1}^{1} U \subset U$. Finally, suppose that U is of the form $\{x \in X | a < x < b\}$. Then we can simply intersect the sets $S_1, S_2 \in \mathcal{S}$ where $S_1 = \{x \in X | a < x\}$ and $S_2 = \{x \in X | x < b\}$ to get that $\bigcap\limits_{n=1}^{2} S_n = \{x \in X | a < x < b\} \subset U$. Thus by condition (2) of Theorem 4.14, we have that $\mathcal{S}$ must be a subbasis for the order topology.

□

Exercise 4.19 Verify that the order topology on $\mathbb{R}$ with the usual $<$ order is the standard topology on $\mathbb{R}$.

Solution: Every set of the order topology is of the form $\{x \in \mathbb{R}|x < a\}$ or $\{x \in \mathbb{R}|a < x\}$ or $\{x \in \mathbb{R}|a < x < b\}$ where $a, b \in \mathbb{R}$.

Consider a point p in a set U of the form of $\{x \in \mathbb{R}|x < a\}$ or $\{x \in \mathbb{R}|a < x\}$. Then observe that $p \in B(p, |a - p|)$ is a ball containing p inside U. By definition, these are therefore open sets in the standard topology on $\mathbb{R}$.

If instead U is of the form $\{x \in \mathbb{R}|a < x < b\}$, then for any $p \in U$ observe that $p \in B(p, \min\{p - a, b - p\})$, so that U must also be open in the standard topology on $\mathbb{R}$.

Finally observe that every open set in the standard topology on $\mathbb{R}$ is of the form of a set in the order topology. This is because every open set in $\mathbb{R}_{\text{std}}$ can be bounded from either one or both ends, both possibilities which are captured by elements of the standard topology on $\mathbb{R}$. Thus we can conclude that the order topology on $\mathbb{R}$ with the usual $<$ order is the standard topology on $\mathbb{R}$.

$\square$

Exercise 4.20
Draw pictures of various open sets in the lexigraphically ordered square.

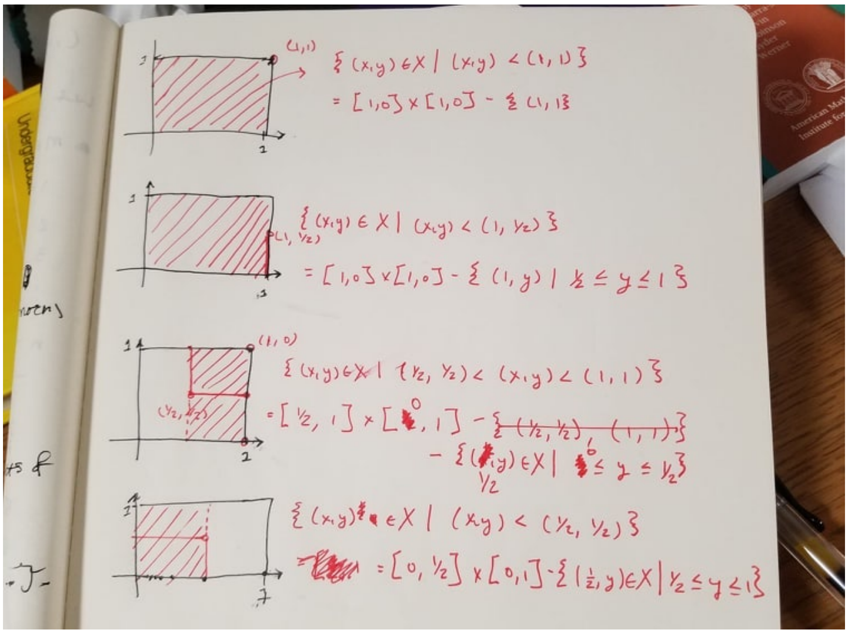


Figure 2: Here we consider three open sets $\{(x,y) \in X|(x,y) < 1\}$, $\{(x,y) \in X|(x,y) < (1,1/2)\}$, $\{(x,y) \in X|(1/2,1/2) < (x,y) < (1,1)\}$ and $\{(x,y) \in X|(x,y) < (1/2,1/2)\}$ and sketch their drawings. Since these are technically basis elements, we can also imagine unioning these sets around to obtain new open sets to imagine what the topology looks like.

4.21 In the lexigraphically ordered square find the closures of the following subsets:

$$A = \left\{ \left(\frac{1}{n}, 0 \right) | n \in \mathbb{N} \right\}$$
$$B = \left\{ \left(1 - \frac{1}{n}, \frac{1}{2} \right) | n \in \mathbb{N} \right\}$$
$$C = \{(x,0) \, | 0 < x < 1\}$$
$$D = \left\{ \left(x, \frac{1}{2} \right) | 0 < x < 1 \right\}$$
$$E = \left\{ \left(\frac{1}{2}, y \right) | 0 < y < 1 \right\}.$$

Solution: For set A, we argue that the set of limit points or the set A is simply the point $(0,1)$. This is because for any open interval containing $(0,1)$, we must have the set wrap back around and include a set of points $(x,0)$ where $x > 0$ and is very small. Since the sequence $\{\frac{1}{n}\}$ converges to 0, we see that an open set $(0,1)$ must include points of the sequence. Therefore (0,1) is a limit point.

For set B, observe that there are no limit points of B. The only possible limit point would be $(1, \frac{1}{2})$, but we can find an open set containing this point but no point of B. For example, the set $\{(x, y) \in X | (1, 0) < (x, y)\}$ contains $(1, \frac{1}{2})$ but not any point of the set B.

For the set C, we can see that the set of limit points will simply be the set $\{(x, 1) \in X | 0 \leq x < 1\} \cup \{(1, 0)\}$. Basically, the the closure is the top and bottom lines of the unit square, minus the two points $(0, 0)$ and $(1, 1)$.

The set D has no limit points. This is because for any point $p = (a, b)$, we can simply construct an open neighborhood about p given by

$$\{(a, y) \in X || y - b| < \min\{|b - \frac{1}{2}|, b - 1\}\}$$

which does not intersect the set if $b \neq \frac{1}{2}$. In the case where $b = \frac{1}{2}$, any neighborhood of the form

$$U = \{(x, y) \in X | (a, 0) < (x, y) < (a, 1)\}$$

contains p, but $(U - \{p\}) \cap D = \emptyset$. Therefore, there are no limit points to the set so $\overline{D} = D$.

For the set E, we see that the set of limit points is simply $\{(\frac{1}{2}, 0), (\frac{1}{2}, 1)\}$. This is because any open set containing either of these points must definitely contain points of the set E. Thus the closure is $E \cup \{(\frac{1}{2}, 0), (\frac{1}{2}, 1)\}$

$\square$

Theorem 4.25 Let $(X, \mathcal{T})$ be a topological space and $Y \subset X$. Then the collection of sets $\mathcal{T}_Y$ is in fact a topology on Y.

Proof: We can show that $\mathcal{T}_Y$ is a topology by showing that it satisfies the four criteria for the definition of a topology.

1. Since $\mathcal{T}_Y = \{U | U = V \cap Y \text{ where } V \in \mathcal{T}\}$, we can take $V = \emptyset$ to observe that $\emptyset \in \mathcal{T}_Y$.

2. Next, since $X \in \mathcal{T}$, we can take $V = X$ to observe that $X \cap Y = Y \in \mathcal{T}_Y$.

3. Suppose $U, V \in \mathcal{T}_Y$ and consider $U \cap V$. Since $U = U' \cap Y$ and $V = V' \cap Y$ for some $U', V' \in \mathcal{T}$, we have that $U \cap V = (U' \cap Y) \cap (V' \cap Y) = (U' \cap V') \cap Y$. Since $(U' \cap V') \in \mathcal{T}$, then we know that $(U' \cap V') \cap Y = (U \cap V) \in \mathcal{T}_Y$. Thus $\mathcal{T}_Y$ is closed under finite intersections.

4. Now suppose $U_\alpha \in \mathcal{T}_Y$ for all $\alpha \in \lambda$, where λ is an arbitrary index. Then for each α there must exist a $V_\alpha \in \mathcal{T}$ such that $U_\alpha = V_\alpha \cap Y$. Thus $\bigcup\limits_{\alpha\in\lambda} U_\alpha = \bigcup\limits_{\alpha\in\lambda} (V_\alpha \cap Y) = \bigcup\limits_{\alpha\in\lambda} (V_\alpha) \cap Y$. Since $\bigcup\limits_{\alpha\in\lambda} V_\alpha \in \mathcal{T}$, we know that $\bigcup\limits_{\alpha\in\lambda} V_\alpha \cap Y \in \mathcal{T}_Y \implies \bigcup\limits_{\alpha\in\lambda} U_\alpha \in \mathcal{T}_Y$. Thus we have that arbitrary unions of open sets are contained in $\mathcal{T}_Y$.

Thus, we have that $\mathcal{T}_Y$ is a topology on Y.

■

Exercise 4.26 Consider $Y = [0, 1)$ as a subspace for $\mathbb{R}_{\text{std}}$ In Y, is the set $[1/2, 1)$ open, closed, neither or both?

Solution: The set $[1/2, 1)$ is not open but is closed under this topology. Firstly, there does not exist an element $V \in \mathbb{R}_{\text{std}}$ such that $[1/2, 1) = V \cap Y$. Thus $[1/2, 1)$ is not open in $\mathcal{T}_Y$. However, it is a closed set since it contains its only limit point $1/2$. This is because every open set which contains $1/2$ must intersect with $[1/2, 1)$. Thus $[1/2, 1)$ is closed in the Y subspace topology.

□

Exercise 4.27 Consider a subspace Y of a topological space X. Is every subset $U \subset Y$ that is open in Y also open in X?

Solution: Consider an open set $U \in \mathcal{T}_Y$. Then there exists a $V \in \mathcal{T}_X$ such that $U = V \cap Y$. While the set U is then technically open in $\mathcal{T}_Y$, it is possible that $V \cap Y$ is not an open set in $\mathcal{T}_X$, which would only happen in $Y \notin \mathcal{T}_X$. Thus we must have that Y be open in order for the above statement to be true.

□

Theorem 4.28 Let $(Y, \mathcal{T}_Y)$ be a subsapce of $(X, \mathcal{T})$. A subset $C \subset Y$ is closed in $(Y, \mathcal{T}_Y)$ if and only if there is a set $D \subset X$, closed in $(X, \mathcal{T})$, such that $C = D \cap Y$.

Proof: Let us first prove the forward direction. Suppose $D \subset X$ is closed in $(X, \mathcal{T})$ and let $C = D \cap Y$. Since D is closed in $(X, \mathcal{T})$, we have that $X - D$ is open in the same topology. Now since $X - D \in \mathcal{T}$, observe that $(X - D) \cap Y \in (Y, \mathcal{T}_Y)$ by definition. Since $(X - D) \cap Y$ is open in $(Y, \mathcal{T}_Y)$, the complement $Y - ((X - D) \cap Y)$ is closed. However, observe that $Y - ((X - D) \cap Y) = Y - (Y - D) = D \cap Y = C$, so that we have concluded that C is a closed set.

Now we prove the other direction. Suppose that C is closed in $(Y, \mathcal{T}_Y)$. Since C is closed, $Y - C$ is open in $(Y, \mathcal{T}_Y)$. Thus by definition, there exists a set $A \in \mathcal{T}_X$ such that $A \cap Y = Y - C$. Since A is open, we know that $X - A$ is closed in $(X, \mathcal{T})$. Call this set D. Now observe that

$$D \cap Y = (X - A) \cap Y = (Y - A) \cap Y = C$$

because $A \cap Y = Y - C$. Thus we have found a set D which closed in $(X, \mathcal{T})$ such that $C = D \cap Y$. Having proved both direction, this proves the theorem.

■

Corollary 4.29 Let $(Y, \mathcal{T}_Y)$ be a subspace $(X, \mathcal{T})$. A subset $C \subset Y$ is closed in $(Y, \mathcal{T}_Y)$ if and only if $\text{Cl}_X(C) \cap Y = C$

Proof: First we'll prove the forward direction. Suppose that $\text{Cl}_X(C) \cap Y = C$. Then let p be a limit point of C in the topological space $(Y, \mathcal{T}_Y)$. Then for every set U open in $(Y, \mathcal{T}_Y)$ which contains p we have that $(U - \{p\}) \cap C \neq \emptyset$. Since U is an open set, there exists a set $V \in \mathcal{T}_X$ such that $U = V \cap Y$. Thus $(V - \{p\}) \cap C \neq \emptyset$ for all open sets $V \in \mathcal{T}_X$ which contain p. This implies that $p \in \text{Cl}_X(C)$. But $p \in \text{Cl}_X(C) \cap Y = C$, so that C contains all of its limit points in $(Y, \mathcal{T}_Y)$. Thus C is closed in $(Y, \mathcal{T}_Y)$.

Now let us prove from the other direction. Suppose that C is closed in $(Y, \mathcal{T}_Y)$. Then C contains all of its limit points in this topological space. We previously showed that all of the limit points in C in $(Y, \mathcal{T}_Y)$ must be in $\text{Cl}_X(C)$. And since $C \subset Y$, we must have that $\text{Cl}_X(C) \cap Y = C$, which is what we set out to show in this direction. Having proven both directions, this proves the corollary.

■

Exercise 4.31 Consider the following subspaces of the lexicographically ordered square.
1. $D = \{(x, \frac{1}{2}) | 0 < x < 1\}$
2. $E = \{(\frac{1}{2}, y) | 0 < y < 1\}$
3. $F = \{(x, 1) | 0 < x < 1\}$.

As sets they are all lines. Describe their relative topologies, especially noting any connections to topologies you have seen already.

Solution:

1. For this set, the relative topology establishes that open sets are simply subsets of the line D. This is because we can imagine creating open sets in the lexicographically ordered squared and intersecting them with out line segment to contain either a point or a subset of D. Thus all subsets of D are open sets, which is similar to the discrete topology we encountered earlier.

2. The relative topology established by this set only contains the empty set and the set E itself, which mirrors the indiscrete topology. We arrive at this conclusion by the fact that intersecting this set with an open set simply yields either the empty set or E itself.

3. By the definition of the relative topology, we see that

$$\mathfrak{T}_F = \{U : U = F \cap V, V \in \mathfrak{T}_{\text{sq}}\}$$

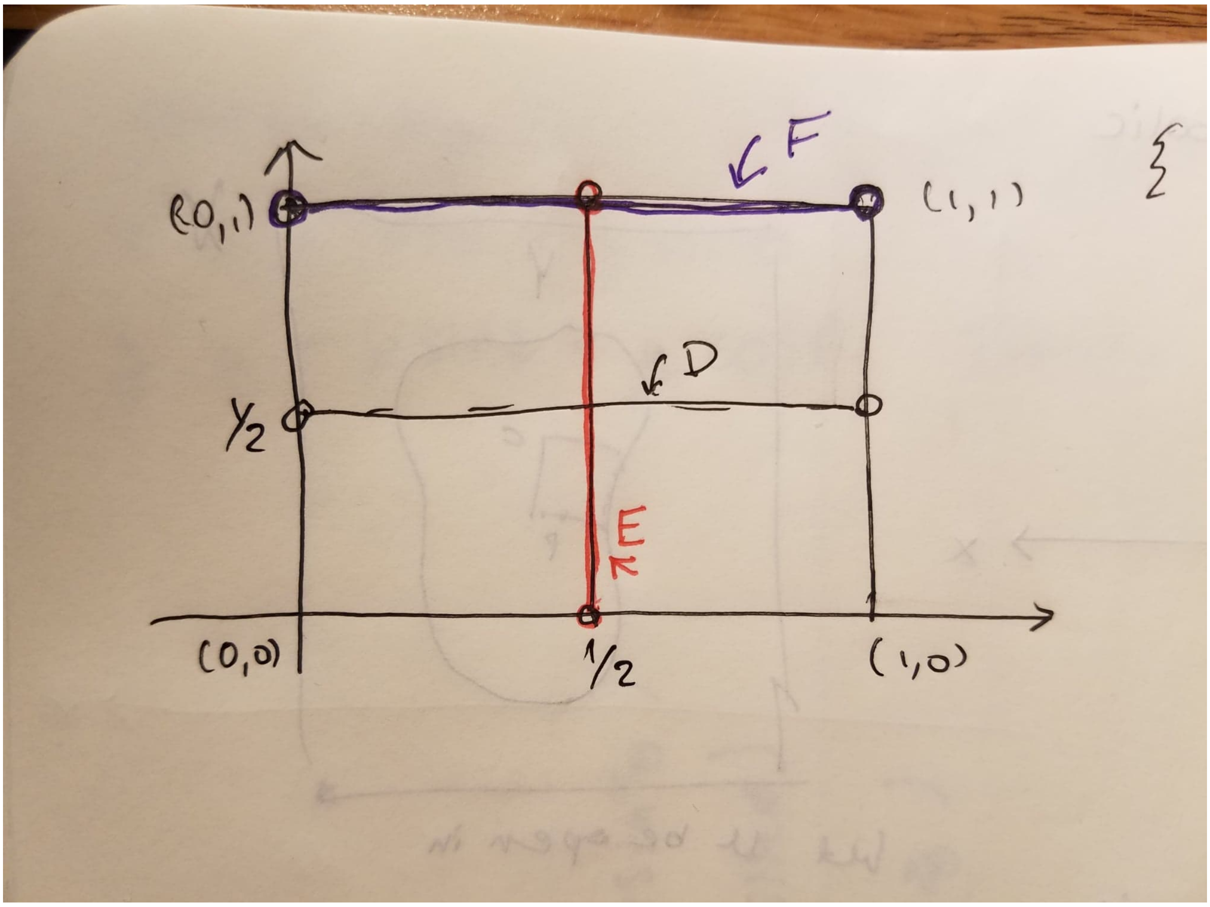
where $\mathfrak{T}_{\text{sq}}$ is the topology of the lexicographically ordered square. If $V \in \mathfrak{T}_{\text{sq}}$ and $V \subset F$, then obviously $V \in \mathfrak{T}_F$. Thus we suspect that subsets of F will be in $\mathfrak{T}_F$. In our case, the subsets of F include the empty set, intervals of the form $a < x < b$, $a \leq x < b$, $a \leq x \leq b$, and $a \leq x \leq b$ where $a, b \in (0, 1)$. However, the last two forms are not allowed in the topology, since the rightmost endpoints are not included. Because of this, we see that $\mathfrak{T}_F$ has a connection with $\mathbb{R}_{LL}$, whose topology consists of intervals where the left hand point is inclusive but the righthand point is not inclusive.

□

Exercise 4.32 Verify that the collection of basic open sets above satisfies the conditions of Theorem 4.3, thus confirming that this collection is a basis for a topology.

Solution: First observe that the first condition of Theorem 4.3 is satisfied, since for any $(p, q) \in X \times Y$ there exist open sets $U \in X$ and $V \in Y$ such that $p \in U$ and $q \in V$. Thus there exists a basic open set $U \times V$ such that $(p, q) \in U \times V$, which shows that each point of $X \times Y$ is in some basic open set.

Now suppose U, V are basic open sets. Then $U = A \times B$ and $V = C \times D$ for some open sets $A, C \in \mathfrak{T}_X$ and $B, D \in \mathfrak{T}_Y$. Let $p \in U \cap V = (A \cap C) \times (B \cap D)$. Then observe that $(A \cap C) \in \mathfrak{T}_X$ and $(B \cap D) \in \mathfrak{T}_Y$. Since the basis consists of the product of all open sets in X and all open sets in Y, we see that $(A \cap C) \times (B \cap D)$ must be a basic open set. Thus we have a basis element $W = (A \cap C) \times (B \cap D)$ such that $p \in W \subset U \cap V$,



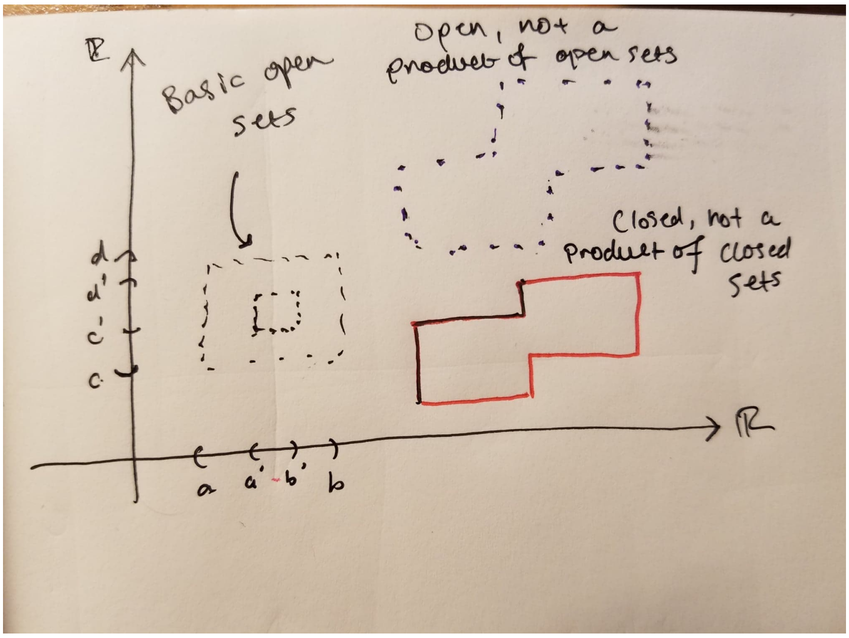
which satisfies the second part of Theorem 4.3. Thus the proposed collection is in fact a basis for the topology.

□

Exercise 4.33 Draw examples of basic and arbitrary open sets in $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ using the standard topology on $\mathbb{R}$. Find (i) an open set in $\mathbb{R} \times \mathbb{R}$ that is not the product of open sets, and (ii) a closed set in $\mathbb{R} \times \mathbb{R}$ that is not the product of closed sets.

Exercise 4.34 Is the product of closed sets closed?

Solution: Yes. Let $p \in \overline{U}$ and $q \in \overline{V}$. Then for every open set U_p containing p and V_q containing q, we'll have that $U_p \cap U \neq \emptyset$ and $V_q \cap V \neq \emptyset$. Therefore we see that $(U_p \times U_q) \cap (U \times V) \neq \emptyset$, meaning that $(p, q) \in \overline{U \times V}$. Thus we have that $\overline{U} \times \overline{V} \subset \overline{U \times V}$.



Now suppose $(p,q) \in \overline{U \times V}$. Then every basic open set of the form $U' \times V'$ containing (p,q) intersects $U \times V$. In other words, for every open set $U' \in \mathcal{T}_X$ containing p, then $U' \cap U \neq \emptyset$. Similarly, for every open $V' \in \mathcal{T}_Y$ containing q, $V' \cap V \neq \emptyset$ Thus we must have that $p \in \overline{U}$ and $q \in \overline{V}$, so that $(p,q) \in \overline{U} \times \overline{V}$, which implies that $\overline{U \times V} \subset \overline{U} \times \overline{V}$.

Since $\overline{U \times V} \subset \overline{U} \times \overline{V}$ and $\overline{U} \times \overline{V} \subset \overline{U \times V}$, we have that $\overline{U \times V} = \overline{U} \times \overline{V}$. Thus if U, V are closed, $\overline{U} = U$ and $\overline{V} = V$, so $U \times V = \overline{U \times V}$. So the product of closed sets is in fact closed.

□

Exercise 4.35 Show that the product topology $X \times Y$ is the same as the topology generated by the subbasis of inverse images of open sets under the projection functions, that is the subbasis is $\{\pi_X^{-1}(U) | U \text{ is open in } X\} \cup \{\pi_Y^{-1}(V) | V \text{ is open in } Y\}$.

Solution: Let U be open. Then for $p \in U$, there exists a basic open set $W \in \mathcal{T}_{\text{prod}}$, the product topology, such that

$$p \in W \subset U.$$

Observe that $W = W_x \times W_y$ where $W_x \in \mathcal{T}_x$ and $W_y \in \mathcal{T}_y$. Note also that

$$W = W_x \times W_y = (W_x \times Y) \cap (X \times W_y).$$

Let $\mathcal{S}$ be the set of inverse images of open sets under the projection functions. Then we also know that $\pi_X^{-1}(W_x) = W_x \times Y$, while $\pi_X^{-1}(W_y) = X \times W_y$, which are in $\mathcal{S}$. We can then state that

$$W = \pi_X^{-1}(W_x) \cap \pi_Y^{-1}(W_y).$$

Therefore, we see that

$$p \in \pi_X^{-1}(W_x) \cap \pi_Y^{-1}(W_y) \subset W.$$

Since for each $V \in \mathcal{S}$ we have that $V \in \mathcal{T}_{\text{prod}}$, (1) $\mathcal{S} \subset \mathcal{T}_{\text{prod}}$ and (2) for any open set U and point $p \in W$ there exists elements $\pi_X^{-1}(W_x), \pi_X^{-1}(W_y) \in \mathcal{S}$ such that $p \in \pi_X^{-1}(W_x) \cap \pi_Y^{-1}(W_y) \subset U$, we have by Theorem 4.14 that $\mathcal{S}$ is a subbasis of the product topology, as desired.

□

Exercise 4.36 Using the standard topology on $\mathbb{R}$, is the product topology $\mathbb{R} \times \mathbb{R}$ the same as the standard topology on $\mathbb{R}^2$?

Solution: Consider $B(p, R)$, a disk of radius R centered at $p = (p_x, p_y)$, which is an open set in $\mathbb{R}_{\text{std}}$.

Observe that for each $q = (q_x, q_y) \in B(p, R)$, we can construct a set $U_q = (q_x - \epsilon, q_x + \epsilon)$ containing q_x if

$$p_x - R < q_x - \epsilon \quad q_x + \epsilon < p_x + R$$

and similarly we can for the set $V_q = (q_y - \delta, q_y + \delta)$ containing q_y if

$$p_y - R < q_y - \delta \quad q_y + \epsilon < p_y + R.$$

Therefore, $q \in U \times V \subset B(p, R)$. Since for each $q \in B(p, R)$ we can find an open $W_p = U_q \times V_q$ such that $p \in W_q \subset B(p, R)$, we see that

$$\bigcup_{q \in B(p,R)} W_q = B(p, R).$$

Thus we see that the product topology is a subset of the standard topology on $\mathbb{R}$. Now consider a basic open set $W = U \times V$ in the product topology on $\mathbb{R} \times \mathbb{R}$. Thus $U = (a, b)$ and (c, d), where a, b, c, d may or may not be finite.

Observe that for any $p = (p_x, p_y) \in U \times V$, we can contain it in a ball $B(p, \epsilon)$ where

$$\epsilon < \min\{\min\{b - p_x, p_x - a\}, \min\{c - p_y, p_y - c\}\}.$$

Therefore, we see that for any $p \in U \times V$ there exists an open ball $B(p, \epsilon)$ such that $p \in B(p, \epsilon) \subset U \times V$. Hence

$$\bigcup_{p \in U \times V} B(p, \epsilon_p) = U \times V$$

so that the standard topology is a subset of the product topology. Since we show the converse, we must have that the standard topology is equivalent to the product topology on $\mathbb{R}$.

□

Exercise 4.37 A basis for the product topology on $\prod_{\alpha \in \lambda} X_\alpha$ is the collection of all sets of the form $\prod_{\alpha \in \lambda} U_\alpha$ where U_α is open in X_α for each α and $U_\alpha = X_\alpha$ for all but finitely many α.

Solution: Consider an open set U in the product topology $\mathcal{T}_{\text{prod}}$. Then for each $p \in U$, there exists a subbasic open set in $\mathcal{S}$ such that

$$p \in \bigcap_{i=1}^{n} \pi_{\alpha_i}^{-1}(U_{\alpha_i}) \subset U$$

Now consider the family of sets described in the problem, and call this set $\mathcal{T}'_{\text{prod}}$. Observe that we can write

$$\bigcap_{i=1}^{n} \pi_{\alpha_i}^{-1}(U_{\alpha_i}) = \cdots \times U_{\alpha_1} \times \cdots \times U_{\alpha_n} \times \cdots = \prod_{\alpha \in \lambda} U_\alpha$$

where $U_\alpha = X_\alpha$ for all $\alpha \in \lambda \backslash \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ and $U_{\alpha_1}, U_{\alpha_2}, \ldots U_{\alpha_n}$ are all restricted open sets in the spaces $X_{\alpha_1}, X_{\alpha_2}, \ldots X_{\alpha_n}$, respectively. Thus $\mathcal{S} \subset \mathcal{T}'_{\text{prod}}$.

$$p \in \prod_{\alpha \in \lambda} U_\alpha \subset U. \tag{1}$$

Now observe that for any $V \in \mathcal{T}'_{\text{prod}}$, $V = \prod_{\alpha \in \lambda} U_\alpha$ where U_α is open in X_α for each α and $U_\alpha = X_\alpha$ for all but finitely many α,

$$V = \prod_{\alpha \in \lambda} U_\alpha = \cdots \times U_{\alpha_1} \times \cdots \times U_{\alpha_n} \times \cdots = \bigcap_{i=1}^{n} \pi_{\alpha_i}^{-1}(U_{\alpha_i})$$

Thus we see that $\mathcal{T}'_{\text{prod}} \subset \mathcal{S}$, so that these two collections of sets generate the same topology: namely, the product topology. More specifically, we see that (1) $\mathcal{T}'_{\text{prod}} \subset \mathcal{T}_{\text{prod}}$ and (2) equation (1) satisfies Theorem 4.3, which proves that $\mathcal{T}_{\text{prod}}$ forms a basis for the product topology, as desired.

□

Exercise 4.38 Let $\mathcal{T}$ be the topology on 2^X with basis generated by the subbasis $\mathcal{S}$.

1. Every basic open set in 2^X is both open and closed.
2. Show that if a collection of subbasic open sets of 2^X has the property that every point of 2^X lies in at least one of those subbasic open sets, then there are two subbasic open sets in that collection such that every point of 2^X lies in one of those two subbasic open sets.
3. Show that if a collection of basic open sets of 2^X has the property that every point of 2^X lies in at least one of those basic open sets, then there are a finite number of basic open sets in that collection such that every point of 2^X lies in one of those basic sets.

Solution:

1. Consider an arbitrary basic open set U in the product topology of 2^X. Then observe that U is of the form

$$U = \{f \in 2^X : f(a_1) = \delta_1, f(a_2) = \delta_2, \ldots, f(a_n) = \delta_n\}$$

where $a_1, \ldots, a_n \in A$ and $\delta_1, \delta_2, \ldots, \delta_n \in \{0, 1\}$. Then observe that

$$U^c = \{f \in 2^X : f(a_1) = |\delta_1 - 1|, f(a_2) = |\delta_2 - 1|, \ldots, f(a_n) = |\delta_n - 1|\}$$

Since U is open, U^c is closed. However, U^c is still of the form of basic open set, which means that U is closed. Therefore, every basic open set in 2^X is open and closed.

2. Let our subbasic open cover be $\{U_\alpha\}_{\alpha\in\lambda}$ where $U_\alpha \in \mathcal{T}$ for all $\alpha \in \lambda$. Now suppose there aren't two subbasic open sets such that every point of 2^X lies in one or the other. Then observe that this is not a cover of 2^X since, if a point of 2^X lies in one set U, then it does not lie in U^c. Thus in this case it would not even be a cover.

3. Let $\{U_\alpha\}_{\alpha\in\lambda}$ be our cover as previously defined. Fix $p \in 2^X$, and observe it lives in some subbasic set

$$U = \{f \in 2^X : f(a_1) = \delta_1, f(a_2) = \delta_2, \ldots, f(a_n) = \delta_n\}.$$

Observe that every point of x either lies in this set, or its coordinate values $f(a_1), f(a_2), \ldots, f(a_n)$ differ in at least one coordinate from the restriction offered by U. Since every coordinate can have at most 2 different values, we see that there are 2^n different ways that the coordinate values $f(a_1), f(a_2), \ldots, f(a_n)$ could differ from the restriction offered in U. Thus we can have at most $2^n + 1$ basic open sets which contain all the points in X, which means that there are at most a finite number of basic open sets that cover every point of 2^X.

□

Exercise 4.39 In the product space $2^{\mathbb{R}}$, what is the closure of the set Z consisting of all elements of $2^{\mathbb{R}}$ that are 0 on every rational coordinate, but may be 0 or 1 on any irrational coordinate? Equivalently, thinking of $2^{\mathbb{R}}$ as subsets of $\mathbb{R}$, what is the closure of the set Z consisting of all subsets of $\mathbb{R}$ that do not contain any rational?

Solution: Observe that

$$Z^c = \bigcup_{n \in \mathbb{N}}^{\infty} \{f : 2^{\mathbb{R}} | f(a_1) = f(a_2) = \cdots = f(a_n) = 0, a_1, a_2, \ldots, a_n \in \mathbb{Q}\}$$

where $a_1, a_2, \ldots, a_n$ are distinct but arbitrarily chosen of $\mathbb{Q}$. Note that this is an uncountable union of open sets, since we are only making a finite number of restrictions on the coordinates. Therefore Z must be closed, so $\overline{Z} = Z$.

□

Exercise 4.40 Find a subset A of $2^{\mathbb{R}}$ and a limit point x of A such that no sequence in A converges to x. For an ever greater challenge, determine whether you can find such an example if A is countable.

Solution: Observe that the point $p \in 2^{\mathbb{R}}$ such that

$$p(a) = 1 \quad \forall a \in \{\pi + q : q \in \mathbb{Q}\}$$
$$p(a) = 0 \quad \forall a \in \mathbb{R} - \{\pi + q : q \in \mathbb{Q}\}$$

is a limit point of Z, the set consisting of all elements of $2^{\mathbb{R}}$ that are 0 on every rational coordinate and 0 or 1 on all others. (Note: π was chosen randomly. We could have done it with any other irrational. All we want is a point such that it's x-th coordinate is 1 for a countable number of irrational x. We know $\mathbb{Q}$ is countable, so adding π to every element of this set generates a countable set of irrationals, which is how we want to design our point.)

Any open set containing p must be of the form

$$\{f \in 2^{\mathbb{R}} | f(\pi + q_1) = f(\pi + q_2) = \cdots = f(\pi + q_n) = 1, q_1, q_2, \ \ldots, q_n \in \mathbb{Q}\}$$

and hence will intersect Z.

By Theorem 3.30, if there exists a sequence of elements of Z which converge to p then $p \in \overline{Z}$. But in 4.39 we saw $\overline{Z} = Z$, and clearly $p \notin Z$. Hence $p \notin \overline{Z}$, so there is no sequence of elements of Z which converge to p.

□

Exercise 4.41 Let $\mathbb{R}^\omega$ be the countable product of copies of $\mathbb{R}$. So every point in $\mathbb{R}^\omega$ is a sequence $(x_1, x_2, x_3, \ldots)$. Let $A \subset \mathbb{R}^\omega$ be the set consisting of all points with only positive coordinates. Show that in the product topology, $\mathbf{0} = (0, 0, 0, \ldots)$ is a limit point of the set A, and there is a sequence of points in A converging to $\mathbf{0}$. Then show that in the box topology, $\mathbf{0} = (0, 0, 0, \ldots)$ is a limit point of the set A, but there is no sequence of points in A converging to $\mathbf{0}$.

Solution: Let U be an open set containing $\mathbf{0}$. Suppose $\mathbf{0}$ is in the basic open set B of the product topology so that $B = \Pi_{\alpha \in \omega} U_\alpha$ where U_α is open in $\mathbb{R}$, and $U_\alpha = \mathbb{R}$ for all but finitely many α. Then for each $\alpha \in \omega$ corresponding to $U_\alpha \neq \mathbb{R}$, the open set must contain $\mathbf{0}$, and hence it must contain positive points of $\mathbb{R}$. Since the rest of the U_α's such that $U_\alpha = \mathbb{R}$ obviously contain positive coordinates of $\mathbb{R}$, we see that the basic open set, and hence the set $U - \{\mathbf{0}\}$, must have a nonempty intersection with A. Therefore $\mathbf{0}$ is a limit point of A in the product topology.

I claim that the sequence $(\frac{1}{n}, \frac{1}{n}, \ldots)$ is a sequence which converges to $\mathbf{0}$ in the product topology. Observe that we can contain $\mathbf{0}$ in a basic open set B, where again $B = \Pi_{\alpha \in \omega} U_\alpha$ and U_α is open in $\mathbb{R}$ while $U_\alpha = \mathbb{R}$ for all but finitely many α. Thus for each $U_\alpha \neq \mathbb{R}$, let $n_\alpha \in \mathbb{N}$ be such that $\frac{1}{n_\alpha} \in U_\alpha$. Now let

$$n = \min \left\{ n_\alpha \middle| \frac{1}{n_\alpha} \in U_\alpha \right\}.$$

Then for $i > n$, we see that $(\frac{1}{i}, \frac{1}{i}, \ldots) \in B$. Thus every open set about $\mathbf{0}$ will contain points of the sequence, which shows that this sequence converges to $\mathbf{0}$ in the product topology.

Now we'll show that there is no sequence which converges to $\mathbf{0}$ in the box topology. Suppose for the sake of contradiction that there is a sequence of points

$$\begin{matrix}(a_{11} & a_{12} & a_{13} & \dots) \\ (a_{21} & a_{22} & a_{23} & \dots) \\ (a_{31} & a_{32} & a_{33} & \dots)\end{matrix}$$

which converge to $\mathbf{0}$. Then observe that we can construct an open set about $\mathbf{0}$ in the box topology as follows. Let (a_1, a_2) contain 0 but exclude (a_{11}). Let (a_3, a_4) contain $\mathbf{0}$ but exclude (a_{22}). If we continue in this fashion, we'll construct an open set in the box topology

$$(a_1, a_2) \times (a_3, a_4) \times \dots$$

which all contain $\mathbf{0}$ but exclude every point of the proposed sequence. The fact that we can create this open set containing $\mathbf{0}$ but no element of the sequence contradicts our claim, which shows that no sequence in the box topology can converge to $\mathbf{0}$.

$\square$

Exercise 4.42 Show that the set $2^{\mathbb{N}}$ in the box topology is a discrete space, whereas the set $2^{\mathbb{N}}$ in the product topology has no isolated points.

Solution: Observe that we can think of an open set here under the box topology as a a set of points where we are allowed to make an infinite number of restrictions on each coordinate. With this perspective, it is then clear that every point is a basic open set, since every point p has a restriction on every single coordinate. Since every point is open, we have that all subsets of $2^{\mathbb{N}}$ are open, which implies that the set is a discrete space under the box topology.

Consider any basic open set in $2^{\mathbb{N}}$ under the product topology:

$$U = \{f \in 2^{\mathbb{N}} : f(a_1) = \delta_1, \dots, f(a_n) = \delta_n\} \quad \delta_1, \dots, \delta_n \in \{0, 1\}$$

Observe that the set contains the point

$$p = (\dots, \overbrace{\delta_1}^{f(a_1)}, \dots, \overbrace{\delta_2}^{f(a_2)}, \dots, \overbrace{\delta_n}^{f(a_n)}, \dots, \overbrace{\delta_{n+1}}^{\text{unrestricted by } U}, \dots)$$

where $\delta_{n+1} \in \{0, 1\}$. But U also contains another point p' such that

$$p' = (\dots, \overbrace{\delta_1}^{f(a_1)}, \dots, \overbrace{\delta_2}^{f(a_2)}, \dots, \overbrace{\delta_n}^{f(a_n)}, \dots, \overbrace{1 - \delta_{n+1}}^{\text{unrestricted by } U}, \dots).$$

Thus no basic open set in $2^{\mathbb{N}}$ in the product topology contains a single element. Hence, there are no isolated points.

□

5.1 Hausdorff, Regular and Normal Spaces.

In the definition of a T_1 space, isn't it sufficient to simply state that there exists an open set U about x such that $y \notin U$?

Theorem 5.1 A space $(X, \mathcal{T})$ is T_1 if and only if every point of X is closed.

Proof: First we'll prove the forward direction. Suppose every point $x \in X$ is a closed set. Then $X - \{x\}$ is open, so foe $y \in X$, $y \neq x$, there exists an open set U such that $y \in U$ and $U \cap \{x\} = \emptyset \implies x \notin U$. Analogously, $X - \{y\}$ is open so there exists an open set V containing x such that $x \in V$ but $y \notin V$. Since x, y were arbitrary distinct points of X, we have that X is a T_1 space.

Now suppose that X is a T_1 space. Consider an $x \in X$ and suppose for a contradiction that $y \in X$, $y \neq x$ is a limit point of $\{x\}$. Then every open set of y must contain x. However, this is not possible since x and y are distinct points, X is T_1, and therefore there exists an open set U containing x such that $y \notin U$. Thus y can't be a limit point, which means that no element in $X - \{x\}$ is a limit point of $\{x\}$. Therefore, x must be a closed set, and since x was arbitrary this shows that every point of X must be a closed set.

■

Exercise 5.2 Let X be a space with the finite complement topology. Show that X is T_1.

Solution: Observe that $X - \{x\}$ is an open set in the finite complement topology for all $x \in X$. Then its complement, $X - (X - \{x\}) = \{x\}$ is closed. Therefore every point is a closed set, and by the Theorem 5.1 we have that X is a T_1 space.

□

Exercise 5.3 Show that $\mathbb{R}_{\text{std}}$ is Hausdorff.

Solution: Consider any two distinct points x and y in $\mathbb{R}$. Then observe that we can construct the open sets $B(x, \epsilon/2)$ and $B(y, \epsilon/2)$ where $|x - y| < \epsilon$ so that $B(x, \epsilon/2)$ and

$B(y, \epsilon/2)$ are disjoint but $x \in B(x, \epsilon/2)$ and $y \in B(y, \epsilon/2)$. Since x, y were distinct and arbitrary, we have that $\mathbb{R}_{\text{std}}$ is a Hausdorff space.

$\square$

Exercise 5.4 Show that $\mathbb{H}_{\text{bub}}$ is regular.

Solution: We found earlier that all subsets of the x-axis are closed in $\mathbb{H}_{\text{bb}}$. Thus if we have a closed subset A and a point $x \notin U$, we can use exercise 4.10(4) to show that there must exist disjoint open sets U and V such that $x \in U$ and $A \subset V$. Therefore, $\mathbb{H}_{\text{bub}}$ is regular.

$\square$

Exercise 5.5 Show that $\mathbb{R}_{LL}$ is normal.

Solution: Let A, B be two disjoint closed sets. Consider $a \in A$, and observe that $\mathbb{R}_{\text{LL}} - B$ is an open set containing a. Therefore, there exists a basis element $[x, y)$ such that $a \in [x, y) \subset \mathbb{R}_{\text{LL}} - B$. Therefore, $[a, y) \subset [x, y) \subset (\mathbb{R}_{\text{LL}} - B)$. Observe that we can create open sets $[a, y)$ for all $a \in A$. Thus let $U = \bigcup\limits_{a \in A} [a, y)$, which is open as it is the arbitrary union of open sets. Similarly, if we take a $b \in B$ and find a basic open set $[x', y')$ such that $b \in [x', y') \subset \mathbb{R}_{\text{LL}} - A$, then we can define an open set $V = \bigcup\limits_{b \in B} [b, y')$.

Now U and V cannot intersect. Each member $[a, y')$ in the union of U is a subset of $\mathbb{R}_{\text{LL}} - B$, while each member $[b, y')$ in the union of V is a subset of $\mathbb{R}_{\text{LL}} - A$, and if they did intersect then this would require that for some $a \in A$, $b \in B$, $[a, y) \cap [b, y') \neq \emptyset$. However, this is impossible as this would imply that either $b \in [a, y)$ or $a \in [b, y')$, which cannot happen since $[a, y) \subset \mathbb{R}_{\text{LL}} - B$ and $[b, y') \subset \mathbb{R}_{\text{LL}} - A$. Thus we have that $U \cap V = \emptyset$. Since A and B were arbitrary disjoint closed sets in $\mathbb{R}_{\text{LL}}$, we see that X must be normal by the definition of normality.

$\square$

Exercise 5.6 1. Consider $\mathbb{R}^2$ with the standard topology. Let $p \in \mathbb{R}^2$ be a point not in a closed set A. Show that

$$\inf\{d(a, p) | a \in A\} > 0.$$

(Recall that inf E is the greatest lower bound of a set of real numbers E.)

2. Show that $\mathbb{R}^2$ with the standard topology is regular.
3. Find two disjoint closed subsets A and B of $\mathbb{R}^2$ with the standard topology such that

$$\inf\{d(a,b)|a \in A \text{ and } b \in B\} = 0$$

4. Show that $\mathbb{R}^2$ with the standard topology is normal.

Solution:

1. Firstly we know that $\inf\{d(a,p)|a \in A\} \geq 0$ since the distance function is always greater than or equal to zero. Thus we must simply show that it is not zero for any $a \in A$ where $p \notin A$. First, observe that since p is not a limit point of A, there exists an open set $B(p,\epsilon)$ containing p such that $B(p,\epsilon) \cap A = \emptyset$. Therefore, we have that $\inf\{d(a,p)|a \in A\} > \epsilon > 0$, proving the desired result.

2. Let $x \in \mathbb{R}^2$ and suppose A is a closed set not containing x. Since x is not a limit point in A, there exists an open set U containing x such that $U \cap A = \emptyset$. Thus let $B(x,\epsilon) \subset U$. Then if we take each point in $a \in A$ and construct an open ball $B(a,\epsilon/2)$ where $\epsilon = \inf\{d(a,p)|a \in A\}$, then none of these balls intersect U. If we union these set of balls, we'll obtain an open set which contains A but is disjoint with x. Thus by definition, $\mathbb{R}^2$ with the standard topology is regular.

3. Consider the set of points which lies on the line $x = 0$ and $y = \frac{1}{x}$. The function y converges to the y-axis, and while these two are sets are closed and disjoint we see that the inf of their distances between their points converges to 0.

4. If we have two disjoint closed sets A and B, then no point of one set is a limit point of the other. Construct a ball about each point of a given by $B(a,r_a)$ where $r_a = \frac{1}{4}\inf\{d(a,b)|b \in B\}$. By part (a), we know that $r_a > 0$. Similarly, let us constuct balls about each point $b \in B$ of radius $r_b = \frac{1}{4}\inf\{d(a,b)|a \in A\}$ given by $B(b,r_b)$ Now observe that no ball from the set $\{B(a,r_a)|a \in A\}$ intersects with any ball from the set $\{B(b,r_b)|b \in B\}$, and that $A \subset \bigcup\limits_{a\in A} B(a,r_a)$ and $B \subset \bigcup\limits_{b\in B} B(b,r_b)$. Since A and B were arbitrary closed sets, we must have that $\mathbb{R}^2$ is normal.

□

Note: this can be done for all metric spaces, since we didn't necessarily appeal to explicit properties of $\mathbb{R}^2$!

Theorem 5.7 1. A T_2-space (Hausdorff) is a T_1-space.
2. A T_3-space (regular and T_1) is a Hausdorff space, that is a T_2-space.
3. A T_4-space (normal and T_1) is regular and T_1, that is, a T_3-space.

Proof:

1. In a T_2-space, we have that for every x distinct from y of the topological space, there are disjoint open sets U, V such that $x \in U$ and $y \in V$. As an obvious consequence, for each $x \neq y$, there exists open sets U, V such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$. Since this holds for all distinct $x, y \in X$, we can conclude that by defintion X is also a T_1 space.

2. Let x, y be distinct. Since the space is regular, and because $\{x\}$ is a closed set (by T_1), we know that for every y distinct form x, there must exist disjoint open sets U, V such that $\{x\} \subset U$ and $y \in V$. In other words, there exists disjoint open sets such that $x \in U$, $y \in V$. Thus by definition, we have a T_2 space.

3. Observe that since we have a T_1, every point is a closed set. Furthermore, since we have normality, disjoint closed sets may be contained in disjoint open sets. Thus consider a closed set A and a point $x \notin A$. Then since $\{x\}$ and A are disjoint closed sets, we may construct disjoint open sets U, V such that $\{x\} \subset U$ and $A \subset V$. By definition, this shows that X is also a regular space. Since the space is regular, and T_1 by hypothesis, we know that the space must be T_3 as desired.

■

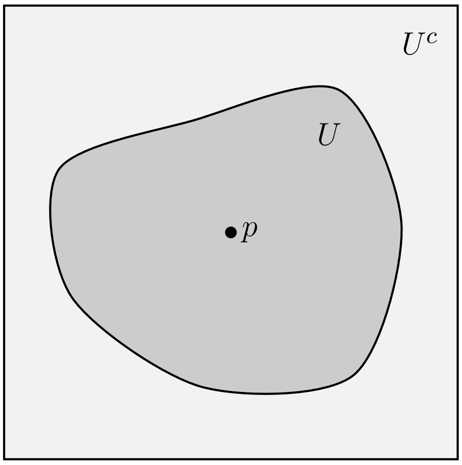
Theorem 5.8 A topological space X is regular if and only if for each point p in X and open set U containing p there exists an open set V such that $p \in V$ and $\overline{V} \subset U$.

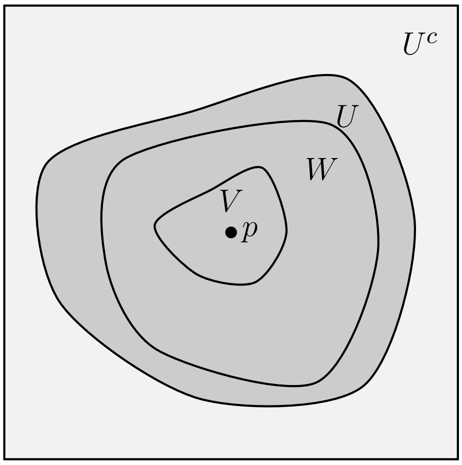
Proof: First we prove the forward direction. Suppose that X is regular and consider some $p \in X$.
Then let $p \in U \subset X$ where U is an open set in X. Observe that U^c is closed and $p \notin U^c$. By regularity, there must exist disjoint open sets V, W such that $p \in V$ and $U^c \subset W$. Now observe that $V \subset W^c$, and since W^c is closed, we know that $\overline{V} \subset W^c$. However, since $U^c \subset W$,

$$U^c \cap W^c = \emptyset \implies U^c \cap \overline{V} = \emptyset \implies \overline{V} \subset U.$$

Thus we have found an open set V such that $p \in V$ and $\overline{V} \subset U$, as desired.

Now we prove the reverse direction. Suppose for each $p \in X$ and open set U containing p, there's an open set V such that $p \in V$ and $\overline{V} \subset U$.
Let A be a closed set not containing $x \in X$. As x is not a limit point of x, there exists an open set U such that $x \in U$ and $U \cap A = \emptyset$.
By hypothesis, there must exist an open V such that $p \in V$ and $\overline{V} \subset U$. Then observe





The first picture shows an arbitrary open set U containing p. In the second picture, we see that $U^c \subset W$, so the boundary of W lives inside U. V is disjoint from W, but contains p, so it also lives inside U.

that (1) $x \in V$ and (2) $A \subset (\overline{V})^c$ and $V \cap (\overline{V})^c = \emptyset$. Since A was an arbitrary closed set and x an arbitrary point not in A, and we contained A and x in disjoint, open sets, we have that X must be a regular space by definition.

■

Theorem 5.9 A topological space X is normal if and only if for each closed set A in X and open set U containing A there exists an open set V such that $A \subset A$ and $\overline{V} \subset U$.

Proof: Suppose that X is normal. Let A be a closed set, and U an open set about A. Since U^c is closed and disjoint from A, there must exist a pair of disjoint open sets V, W such that $U^c \subset V$ and $A \subset W$. Next observe that since V and W are disjoint, we know that $W \subset V^c$. Since V^c is closed, we also know that $\overline{W} \subset V^c$. But $V^c \subset U$; Thus we have that $A \subset W \subset \overline{W} \subset V^c \subset U$. Thus for every closed A and U containing A, there exists an open set W such that $A \subset \overline{W} \subset U$, as desired.

Now suppose that for a closed set A and an open set U containing A there exists an open set V such that $A \subset V$ and $\overline{V} \subset U$. Let B be a closed set which is disjoint from A. Since A and B have no limit points in common, we can see that for each $a \in A$, there exists an open set U_a such that $U_a \cap B = \emptyset$. Thus let $U' = \bigcup_{a \in A} U_a$, which is an open set. Then $U' \cap B = \emptyset$ by construction, and by assumption there must exist an open set W such that $A \subset W$ and $\overline{W} \subset U'$. Next observe that $\overline{U}^c$ is an open set which contains B, so by assumption there exists a W' such that $B \subset W' \subset \overline{W'} \subset \overline{U}^c$. Thus we see that

$A \subset W$ and $B \subset W'$, and $W \cap W' = \emptyset$ since $W \subset U$ but $W' \subset \overline{U}^c$ Since A and B were arbitrary closed sets, and can be contained in disjoint open sets, we have that the space is normal, which proves the assertion.

■

Presented in class 2/20

Theorem 5.10 A topological space X is normal if and only if for each pair of disjoint closed sets A and B there are disjoint open sets U and V such that $A \subset U$, $B \subset V$ and $\overline{U} \cap \overline{V} = \emptyset$.

Proof: First we prove the forward direction. Suppose that X is a normal space. Then for every pair of disjoint closed sets A and B in X, there exist disjoint open sets such that $A \subset U$ and $B \subset V$. However, by Theorem 5.9, we know that there must exist open sets U' and V' such that $A \subset U' \subset \overline{U'} \subset U$ and $B \subset V' \subset \overline{V'} \subset U$. Since U' and V' are disjoint, this proves the existence of disjoint open sets containing A and B whose intersection of their closures is empty.

Next, suppose that for every pair of disjoint closed sets A and B, there are disjoint open sets U and V such that $A \subset U$ and $B \subset V$ and $\overline{U} \cap \overline{V} = \emptyset$. Since A, B are arbitrary disjoint closed sets, $A \subset U$ and $B \subset V$ and $U \cap V = \emptyset$, X satisfies the conditions of a normal space, so X must be normal.

■

Theorem 5.11 (The Incredible Shrinking Theorem.) A topological space X is normal if and only if for each pair of open sets U, V such that $U \cup V = X$, there exist open sets U', V' such that $\overline{U'} \subset U$ and $\overline{V'} \subset V$ and $U' \cup V' = X$.

Proof: First we prove the forward direction. Suppose X is normal and that U, V are open sets such that $U \cup V = X$. Observe that $U^c \subset V$ and $V^c \subset U$. By Theorem 5.9 there must exist open sets U', V' such that

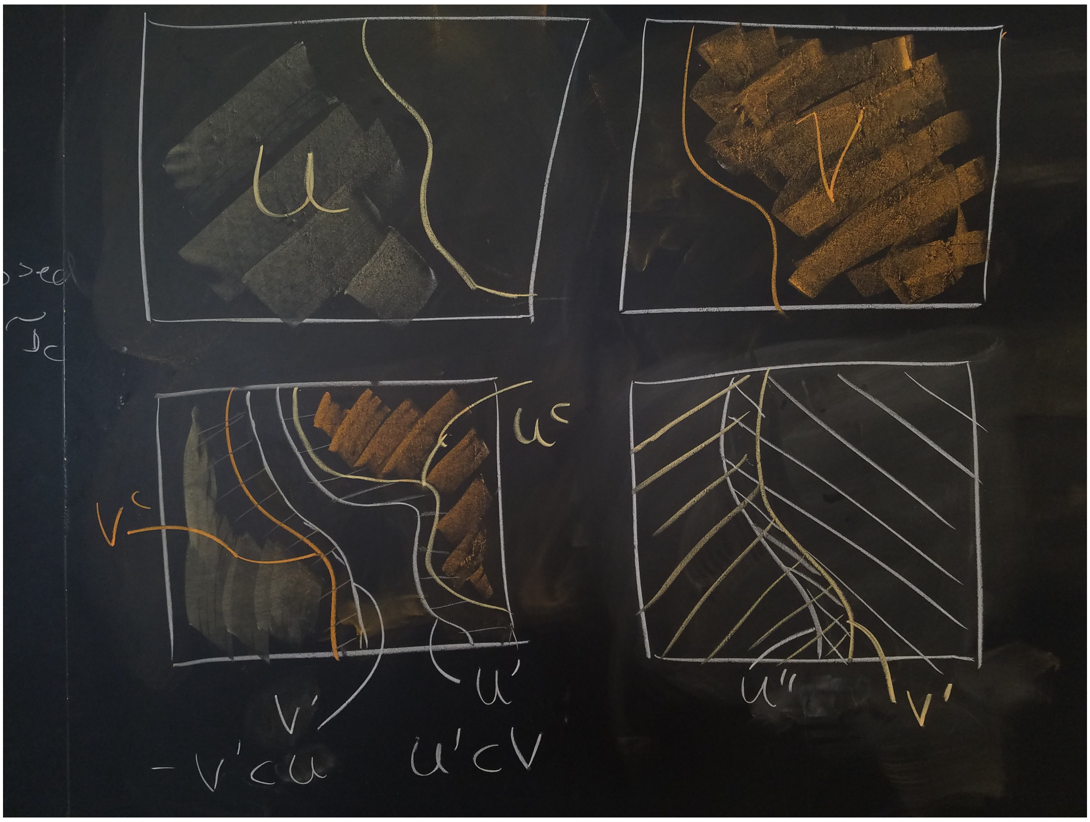
$$U^c \subset U', \overline{U'} \subset V,$$

$$V^c \subset V', \overline{V'} \subset U.$$

Since $(V')^c$ is a closed and $(V')^c \subset V$, we can apply Theorem 5.9 again to conclude that there must exist a set U'' such that

$$(V')^c \subset U'', \overline{U''} \subset V.$$

Now since U'' and V' are open sets such that $\overline{U''} \subset V$, $\overline{V'} \subset U$, and $U'' \cap V' \neq \emptyset$ because $(V')^c \subset U''$, we have that $U'' \cup V' = X$.



Thus this finishes the proof in this direction. Next we prove the other direction. Suppose that for every pair of open sets $U, V \subset X$ such that $U \cup V = X$, there exists open sets U', V' such that $\overline{U'} \subset U$ and $\overline{V'} \subset V$.

Let A and B be disjoint closed sets in X. Observe that A^c, B^c are open sets such that $A^c \cup B^c = X$. Thus there must exist open sets U, V such that

$$\overline{U} \subset A^c \text{ , } \overline{V} \subset B^c, U \cup V = X.$$

Next observe that

$$(A^c)^c \subset (\overline{U})^c \subset U^c \implies A \subset (\overline{U})^c \subset U^c$$
$$(B^c)^c \subset (\overline{V})^c \subset V^c \implies B \subset (\overline{V})^c \subset V^c$$

Since $U \cup V = X$, we have that $U^c \cap V^c = \emptyset$ by DeMorgan's laws. Hence, $(\overline{U})^c$ and $(\overline{V})^c$ are disjoint open sets such that $A \subset (\overline{U})^c$ and $B \subset (\overline{V})^c$. Thus by definition, X is normal.

■

Exercise 5.12

1. Describe an example of a topological space that is T_1 but not T_2.

2. Describe an example of a topological space that is T_2 but not T_3.
3. Describe an example of a topological space that is T_3 but not T_4.

---

1. Finite complement topology is an example. First, every set of the form $X - \{p\}$ where $p \in X$ is open, so the complement $X - (X - \{p\}) = \{p\}$ is closed. Hence by Theorem 5.1, X is T_1.
   Now suppose X is T_2, Then for all $p, q \in X$ and $p \neq q$, there exists disjoint, open sets U, V such that $p \in U$, $q \in V$. However, $U \cap V \implies V \subset U^c$. But this is a contradiction since U^c is finite, by construction, and V must at least be infinite (since we know that V^c is finite.) Thus X is not T_2.

   Another example would be the countable complement topology, and the proof is almost exactly as the one presented for the finite complement topology.

2. The harmonic set is T_2 but not T_3. This is because for any two points $p, q \in \mathbb{R}$ we can contain them in disjoint open sets (a, b) and (c, d) where $a < p < b < c < q < d$ or $c < q < d < a < p < b$. If either p or q are in H, then it is vacuously true that we can contain them in an open set disjoint from any open set containing another point because there are no open sets which contain elements of H.

   The harmonic set is not T_3 since (1) H is a closed set (as it has no limit points) and (2) no open set can contain H. Therefore, it cannot be regular, and hence not T_3.

3. In Exercise 5.4, we showed that $\mathbb{H}_{\text{bub}}$ is regular. Observe that by Exercise 4.10.3 every point on the x-axis can be contained in disjoint open sets, and it is trivial that two distinct points in $\{(x, y) : y > 0\}$ can be contained in disjoint open sets. Thus $\mathbb{H}_{\text{bub}}$ is T_3. However, by Exercise 4.10.5 the rationals and irrationals on the x-axis cannot be contained in disjoint open sets, and the rationals and irrationals are closed sets. Hence $\mathbb{H}_{\text{bub}}$ is not normal. Therefore, $\mathbb{H}_{\text{bub}}$ is T_3 but not T_4.

Exercise 5.14 Show that $\mathbb{H}_{\text{bub}}$ is not normal.

In the previous chapter, we saw that there does not exist disjoint open sets U and V in $\mathbb{H}_{\text{bub}}$ such that $\mathbb{Q} \in U$ and $\mathbb{R} - \mathbb{Q} \subset V$. However, observe that $\mathbb{Q}$ and $\mathbb{R} - \mathbb{Q}$ are closed sets. With this example we see that $\mathbb{H}_{\text{bub}}$ cannot be a normal set.

Theorem 5.15 Order topologies are T_1, Hausdorff, regular and normal.

Proof: (**$\mathbf{T_1}$**.) Suppose X has the order topology. Let $a \in X$. Observe that $\{x \in X | a < x\} \cup \{x \in X\ x > a\}$ is the union of two open sets, so it is open, and hence its complement, which is $\{a\}$, is closed. Thus every singleton set is a closed set, so X is T_1.

(**Hausdorff.**) Consider two distinct points $a, b \in X$, and suppose without loss of generatlity that $a < b$. If there exists an element $c \in X$ such that $a < c < b$ then

$$\{x \in X | x < c\} \text{ and } \{x \in X | c < x\}$$

are two disjoint open sets which contain a, b respectively.
If there is no $c \in X$ such that $a < c < b$, then

$$\{x \in X | x < b\} \text{ and } \{x \in X | a < x\}$$

are two disjoint open sets which contain a, b respectively. Thus X is also a Hausdorff space.

(**Regular.**) Now let A be a closed set and suppose $x \notin A$. Since x is not a limit point of A, we see that there must exist an open set U which contains x and $U \cap A = \emptyset$. Then a open set which is disjoint from U and contains A is $A^c - U$, which shows that the space is regular.

■

Theorem 5.16 Let X and Y be Hausdorff. Then $X \times Y$ is Hausdorff.

Proof: If X and Y are both Hausdorff, then for two distinct $p = (p_x, p_y)$ and $q = (q_x, q_y)$ both in $X \times Y$, there exists disjoint open sets $U_{p_x}, U_{q_x} \in \mathcal{T}_X$ such that $p_x \in U_{p_x}$, $q_x \in U_{q_x}$ and another pair of disjoint sets $V_{p_y}, V_{q_y} \in \mathcal{T}_Y$ such that $p_y \in U_{p_y}$ and $q_y \in U_{q_y}$. Now observe that $p \in U_{p_x} \times U_{p_y}$ and $q \in V_{q_x} \times V_{q_y}$ while $U_{p_x} \times U_{p_y}$ is disjoint with $V_{q_x} \times V_{q_y}$. Since p, q were arbitrary distinct points in $X \times Y$, we have that $X \times Y$ is a Hausdorff space.

■

Theorem 5.17 Let X and Y be regular. Then $X \times Y$ is regular.

Proof: Suppose X and Y are regular, and let $p = (p_x, p_y) \in X \times Y$. Suppose p is contained in an open set W. Then there exists a basic open set of the form $U \times V$ which contains p and where $U \in \mathcal{T}_X$ and $V \in \mathcal{T}_Y$. Now since X and Y are regular, we can use Theorem 5.8 to conclude the existence of open sets U' and V' such that $p_x \in U'$ and $\overline{U'} \subset U$ and $p_y \in V'$ while $\overline{V'} \subset V$. Therefore, $\overline{U'} \times \overline{V'} \subset U \times V$.

Now observe $p \in U' \times V'$ and that, by Exercise 4.34, $\overline{U' \times V'} = \overline{U'} \times \overline{V'} \subset U \times V$ and hence is entirely contained in W. Since p and W were arbitrary, this shows that $X \times Y$ satisfies Theorem 5.8, allowing us to conclude that $X \times Y$ is a regular space.

■

Presented in class 2/25/19

Theorem 5.19 Every Hausdorff is hereditarily Hausdorff.

Proof: Let Y be a subset of X, and consider the relative topology of Y given by

$$\mathcal{T}_Y = \{V | V = Y \cap U, U \in \mathcal{T}_X\}.$$

Since X is Hausdorff, we know that for any distinct pair of points $p, q \in Y$, which are obviously also points in X, there exist disjoint open sets $U', V' \in \mathcal{T}_X$ such that $p \in U'$ and $q \in V'$. Next observe that $U'' = Y \cap U'$ and $V'' = Y \cap V'$ are two disjoint open sets in $\mathcal{T}_Y$ such that $p \in U''$ and $q \in V''$. Since p, q were arbitrary points of Y, we have that Y must also be a Hausdorff space.

■

Theorem 5.20 Every regular space is hereditarily regular.

Proof: Let Y be a subset of X endowed with the relative topology on X. Then consider C closed in $(Y, \mathcal{T}_Y^{\text{rel}})$, and a point $x \in Y$ such that $x \notin C$. From Theorem 4.28, we know that C is closed if and only if there exists a set D closed in $(X, \mathcal{T}_X)$ such that $C = D \cap Y$.

Since X is regular, and we know that $x \notin D$, then for the set D closed in $(X, \mathcal{T}_X)$ there exist disjoint sets $U, V \in (X, \mathcal{T}_X)$ such that $D \subset U$ and $x \in V$. Now observe that $U' = U \cap Y$ and $V' = V \cap Y$ are disjoint sets open in $\mathcal{T}_Y^{\text{rel}}$ such that $C \subset U'$ and $x \in V'$. Since C was an arbitrary closed set in $(Y, \mathcal{T}_Y^{\text{rel}})$ and x was a arbitrary point of Y but not of C, the topological space $(Y, \mathcal{T}_Y^{\text{rel}})$ satisfies the properties of being regular so Y is a regular space.

■

Theorem 5.23 Let A be a closed subset of a normal space X. Then A is normal when given the relative topology.

Proof: Let X be a normal space, and consider the relative topology on A:

$$\mathcal{T}_A^{\text{rel}} = \{U | U = A \cap V \text{ where } V \in \mathcal{T}_X\}$$

Now consider a pair of disjoint closed sets D, C closed in $(A, \mathcal{T}_A^{\text{rel}})$. Then by Theorem

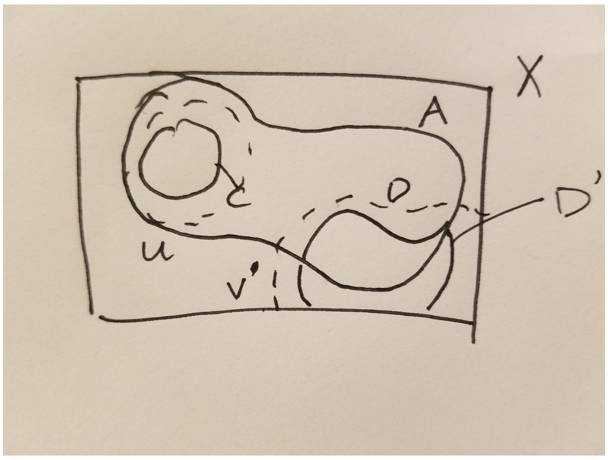


Figure 1: In this figure, we drew a closed set C completely contained in A and a closed set D which shares limit points with A in $(X, \mathcal{T}_X)$.

4.28, we know that there must exist sets D' and C' closed in $(X, \mathcal{T})$ such that $C = A \cap C'$ and $D = A \cap D'$. Now observe that since C and D are result of intersecting two sets which are closed in $(X, \mathcal{T}_X)$, we must have that C and D are also sets closed in $(X, \mathcal{T}_X)$.

Now since X is normal and C and D are disjoint and closed in $(X, \mathcal{T}_X)$, there must exist disjoint, open sets U and V in $(X, \mathcal{T}_X)$ such that $C \subset U$ and $D \subset V$. Next, let $U' = A \cap U$ and $V' = A \cap V$ and observe that (1) U' and V' are disjoint open sets in $(A, \mathcal{T}_A^{\text{rel}})$ and (2) $C \subset U'$ and $D \subset V'$. This is because $C \subset A$, so if $C \subset U$ then we are certain that $C \subset U \cap A = U'$; an identical argument applies to D. Now since C and D were arbitrary closed sets of A, we have that A is a normal space when given the relative topology.

$\blacksquare$

Exercise 5.25 Let Y be a subspace of a topological space X, and let A and B be two disjoint closed subsets of Y in the subspace topology. Show that both $\overline{A} \cap B = \emptyset$ and $A \cap \overline{B} = \emptyset$, where the closures are taken in X.

Solution: If A and B are two disjoint closed sets in the subspace topology, then observe that no point of one set is a limit point of the other. Thus for every point of $a \in A$, we can construct a set $U_a \in \mathcal{T}_Y$ such that $U_a \cap B = \emptyset$. Similarly, for every point $b \in B$ we can construct a set $U_b \in \mathcal{T}_Y$ such that $U_b \cap A = \emptyset$.

Let a be a limit point of A in $(X, \mathcal{T})$, and consider an open set $U \in \mathcal{T}_X$ containing a. Then let $U' = U \cap Y$, and observe that $U' \cap B = \emptyset$ because A and B are disjoint closed sets in $(Y, \mathcal{T}_\mathcal{Y})$. Thus we see that a cannot be a limit point of B, so that $\overline{A} \cap B = \emptyset$, where the closure is taken in X. We can repeat the same argument for B since the argument is symmetric, and conclude that $A \cap \overline{B} = \emptyset$ as well, giving the desired result.

$\square$

Theorem 5.26 The space X is a completely normal space if and only if X is hereditarily normal.

Proof: Suppose X is completely normal. Let Y be a subset and consider any two disjoint, closed sets C and D in the subspace topology

$$\mathcal{T}_Y = \{U | U = V \cap Y : V \in \mathcal{T}_X\}.$$

By Exercise 5.25, these sets are separated in X.

Since C and D are separated in X, there exist two disjoint open sets A, B such that $C \subset A$, $D \subset B$. Therefore, $A \cap Y$ and $B \cap Y$ are two open sets in $\mathcal{T}_Y$ such that $C \subset A \cap Y$ and $B \cap Y$, which shows that Y is normal. Since Y was an arbitrary subset we have that X is hereditarily normal.

Suppose now that X is heredetiarily normal, and consider two separated subsets A and B of X. Denote the subspace $A \cup B$ as Y. Then observe that

$$Y \cap \overline{B} = (A \cup B) \cap \overline{B} = (A \cap \overline{B}) \cup (B \cap \overline{B}) = B.$$

Thus B is closed in Y, since $\overline{B}$ is closed in X, and by Theorem 4.28 this implies that $Y \cap \overline{B} = B$ is a closed set in the subspace Y. By analogous reasoning, we also have that A is closed in Y.

Since X is heredetiarily normal, Y is a subspace, and A, B are disjoint closed sets in Y, we can contain A and B in disjoint open sets U and V in Y. However, we also know $U = U' capY$ and $V = V' \cap Y$ for $U', V' \in \mathcal{T}_X$.

$\blacksquare$

Theorem 5.29 (The Normality Lemma). Let A and B be subsets of a topological space X and let $\{U_i\}_{i\in\mathbb{N}}$ and $\{V_i\}_{i\in\mathbb{N}}$ be two collections of open sets such that

1. $A \subset \bigcup_{i\in\mathbb{N}} U_i$
2. $B \subset \bigcup_{i\in\mathbb{N}} V_i$
3. for each i in $\mathbb{N}$, $\overline{U_i} \cap B = \emptyset$ and $\overline{V_i} \cap A = \emptyset$.

Then there exist open sets U and V such that $A \subset U$, $B \subset V$, and $U \cap V = \emptyset$.

Proof: Suppose that (1) and (2) hold, and let

$$U = \bigcup_{n\in\mathbb{N}} A_n = \bigcup_{n\in\mathbb{N}} \left(U_n - \bigcup_{i=1}^{n} \overline{V_i} \right)$$

and

$$V = \bigcup_{n\in\mathbb{N}} B_n = \bigcup_{n\in\mathbb{N}} \left(V_n - \bigcup_{i=1}^{n} \overline{U_i} \right).$$

Note that U and V are open because they each are the countable union of open sets. This is because each $\bigcup_{i=1}^{n} \overline{U_i}$ and $\bigcup_{i=1}^{n} \overline{V_i}$ finite unions of closed sets, and hence are closed. Thus each $U_n - \bigcup_{i=1}^{n} \overline{V_i}$ and $V_n - \bigcup_{i=1}^{n} \overline{U_i}$ are open sets by Theorem 3.15.

Now observe that $A \subset U$, $B \subset V$. We'll show this is true for A, since the argument that this is true for B will be identical. Thus let $a \in A$. Then $a \in U_n$ for some $n \in \mathbb{N}$. However, $a \notin \overline{V_i}$ for any $i \in \mathbb{N}$, so that $a \in U_n - \bigcup_{i=1}^{n} \overline{V_i}$. Therefore, $a \in \bigcup_{n\in\mathbb{N}} \left(U_n - \bigcup_{i=1}^{n} \overline{V_i} \right) = U$, so that $A \subset U$.

Finally, observe that $U \cap V = \emptyset$. If not, then there exists an $x \in U \cap V$. meaning that for some $m, n \in \mathbb{N}$,

$$x \in U_n - \bigcup_{i=1}^{n} \overline{V_i}$$
$$x \in V_m - \bigcup_{i=1}^{m} \overline{U_i}.$$

Without loss of generality, suppose $n \leq m$. Then by the second equation, we see that $x \notin U_i$ for $i \leq m$. However, this implies that $x \notin U_n$ since $n \leq m$, which contradicts the first above equation. Thus there cannot be such an x, and $U \cap V = \emptyset$.

■

Presented sketch 2/27/18

Theorem 5.30 If X is normal and $C = \cup_{i\in\mathbb{N}} K_i$ is the union of closed sets K_i in X, then the subspace C is normal.

Proof:

■

Theorem 5.31 Suppose a space X is regular and countable. Then X is normal.

Proof: Consider two sets A and B. Since X is regular, we know by an application of the definition that for all $a \in A$, there exist open sets $\{U_a\}$ such that $a \in U_a$ which are each disjoint with $\overline{B}$. Similarly, there must exist open sets $\{U_b\}$ such that $b \in U_b$ which are each disjoint with $\overline{A}$.

Now by Theorem 5.8, we know that for each open set U_a containing a, there exists an open set V_a such that $a \in V_a$ and $\overline{V_a} \subset U_a$. Similarly, for each open set U_b containing b, there exists an open set V_b such that $b \in V_b$ and $\overline{V_b} \subset U_b$

Observe that $A \subset \bigcup\limits_{a\in A} V_a$, $B \subset \bigcup\limits_{b\in B} V_b$, and that $\overline{V_b} \cap A = \overline{V_a} \cap B = \emptyset$ for all $a \in A$ and $b \in B$. Since A, B are at most countable, the sets $\{U_a\}_{a\in A}$ and $\{U_b\}_{b\in B}$ are at most countable.

Thus by the Normality Lemma, we can then conclude there exist open sets U and V such that $A \subset U$ and $B \subset V$ while $U \cap V = \emptyset$. Therefore, we can conclude that X is normal, which is what we set out to show.

■

Presented 2/27/18

Theorem 5.32 Suppose a space X is regular and has a countable basis. Then X is normal.

Proof: Consider two disjoint subsets A and B of X. Since they are disjoint, we know that for each $a \in A$, there exists an open set U_a such that $U_a \cap B = \emptyset$ for all $a \in A$. Similarly for each $b \in B$, we know that there exists an open set $U_b \cap A = \emptyset$ for all $b \in B$.

Observe that the sets $\{U_a\}$ and $\{U_b\}$ may or may not be countable. However, since we have a countable basis $\mathcal{B} = \{B_1, B_2, \ldots\}$, we know by Theorem 4.1 that each $a \in A$ is contained in some basis element B_i such that $a \in B_i \subset U_a$, where $i \in \mathbb{N}$. Thus let

$\mathcal{B}_A$ be the set of basis elements such that $A \subset \bigcup_{B_A \in \mathcal{B}_A} B_A$. Similarly, every $b \in B$ is contained in a basis element B_j such that $b \in B_j \subset U_b$, where $j \in \mathbb{N}$. Now let $\mathcal{B}_B$ by the set of basis elements such that $B \subset \bigcup_{B_B \in \mathcal{B}_B} B_B$.

Now by Theorem 5.8, for each $a \in A$, there exists an open set $V_{j(a)}$ such that $a \in V_{j(a)}$ and $\overline{V_j(a)} \subset B_{i(a)}$ where $j(a) \in \mathbb{N}$ and $i(a) \in \mathbb{N}$ is the index which corresponds to the set in $\{B_1, B_2, \dots\}$ such that $a \in B_{i(a)}$. Similarly for B, we know that for each $b \in B$ there exists an open set $W_{j(b)}$ such that $b \in W_{j(b)}$ where $j \in \mathbb{N}$ and $\overline{W_j(b)} \subset B_{i(b)}$ where $i(b)$ is defined analogously for how we defined $i(a)$.

Finally, observe that $A \subset \bigcup_{j \in \mathbb{N}} V_j$, $B \subset \bigcup_{j \in \mathbb{N}} W_j$, and that $V_j \cap B = W_j \cap A = \emptyset$ for each $j \in \mathbb{N}$. Since these conditions satisfy that the normality lemma, we have that X is normal, which is what we set out to show.

$\blacksquare$

Chapter 6

Countable Features of Spaces: Size Restrictions

Exercise 6.1 Show that A is dense in X if and only if every non-empty open set of X contains a point of A.

Solution: First we prove the forward direction. Suppose that A is a dense subset in X. Then by definition, $\overline{A} = X$. Thus every point of X is a limit point of A, which means that for every point $p \in X$ and every open set U which contains p we see that

$$(U - \{p\}) \cap A \neq \emptyset.$$

Since this holds for all $p \in X$, we see that every open set in X must contain points in A, which proves this direction.

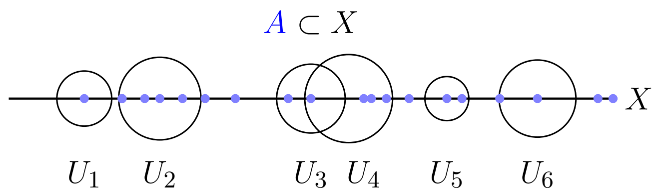


Figure 1:

Here we see the set A is a dense subset in X. The sets $U_1, \dots U_6$ denote arbitrary open sets in X.

Now suppose that every nonempty open set of X contains a point of A. Then this means that for any $p \in X$, any open set U containing p must also contain a point in A. By definition, this is a limit point. Since p was an arbitrary point of x, we must have that every element of X is a limit point of A. Therefore, we must have that $\overline{A} = X$, which finishes the proof in this direction.

$\square$

Exercise 6.2 Show that $\mathbb{R}_{\text{std}}$ is separable. With which of the topologies on $\mathbb{R}$ that you have studied is $\mathbb{R}$ not separable?

Solution: Observe that a countable dense subset in $\mathbb{R}_{\text{std}}$ is the set of rationals. This is because every nonempty open set of $\mathbb{R}$ on the standard topology contains points of $\mathbb{Q}$. By our previous exercise, this allows us to conclude that $\mathbb{Q}$ is dense in $\mathbb{R}$. Since the rationals

are countable, this in total allows us to conclude that $\mathbb{R}_{\text{std}}$ has a countable dense subset, and is therefore separable by definition.
However, this wouldn't hold for the discrete topology on $\mathbb{R}$, since it does not have a countable dense subset with this topology. The countable complement is also not separable, since every open set in the topology must be uncountable and hence finding a countable but dense subset of X is impossible.

□

Exercise 6.4 Find a separable space that contains a subspace that is not separable in the subspace topology.

Solution: Lemma. An uncountable set with the discrete topology is not separable.

Proof. For the sake of contradiction suppose that X is uncountable and is separable under the discrete topology. Then there exists a countable dense set A such that $\overline{A} = X$. However, since X has the discrete topology, we know that $A = \overline{A} = X$; a contradiction since A is countable while X is uncountable. Thus X is not separable under the discrete topology.

Now consider a topology on an uncountable set X given by

$$\mathcal{T} = \{\emptyset\} \cup \{U \subset X\}_{p \in X}$$

where $p \in X$. Observe that $\{p\}$ is dense in this set since every open set in $\mathcal{T}$ contains $\{p\}$ by construction. Since $\{p\}$ is countable and dense, X is separable on this topology.

Consider the subspace $X - \{p\}$. For any $U \subset (X - \{p\})$, we see that $U \cup \{p\} \subset \mathcal{T}$, so that U is open in $X - \{p\}$. Thus every subset of $X - \{p\}$ is open, which implies that this is an uncountable discrete space. However, we know that an uncountable discrete space is not separable, so that $X - \{p\}$ is not separable.

□

Presented in Class ?

Theorem 6.5 If X and Y are separable spaces, then $X \times Y$ is separable.

Proof: Suppose X or Y are separable spaces. Then there exist countable sets A and B such that $\overline{A} = X$ and $\overline{B} = Y$. Using the fact that $\overline{A} \times \overline{B} = \overline{A \times B}$, we see that

$$\overline{A \times B} = \overline{A} \times \overline{B} = X \times Y.$$

Thus $A \times B$ is dense in $X \times Y$. But also observe that $A \times B$ is countable, since we can form a bijection between $A \times B$ and A or B (namely the projection function). Thus $X \times Y$ must be separable because it contains a countable dense subset, which is what we set out to show.

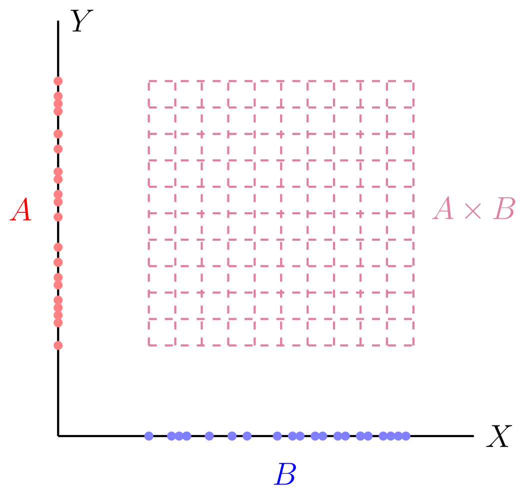


Figure 2: Here in this picture, we see that if A and B are countable dense subsets, then their product forms a countable dense subset.

■

Theorem 6.6 The space $2^{\mathbb{R}}$ is separable.

Proof: Consider

$$A = \left\{f \in 2^{\mathbb{R}} : \bigcup_{i=1}^{n}[p_i, q_i] | p_i, q_i \in \mathbb{Q} \middle| \forall x \in [p,q], f(x) = 1, x \notin [p_i, q_i], f(x) = 0\right\}$$
$$\bigcup \left\{f \in 2^{\mathbb{R}} : \bigcup_{i=1}^{n}[p_i, q_i] | p_i, q_i \in \mathbb{Q} \middle| \forall x \in [p,q], f(x) = 0, x \notin [p_i, q_i], f(x) = 1\right\}$$

that is, we only consider intervals $[p, q]$, which have rational endpoints, and finitely union them. To construct the points f in our set, we assign either a 1 or a 0 to $f(x)$ when x lies in any of the finite intervals $[p_i, q_i]$. (This actually doesn't have to be done with $\mathbb{Q}$, but rather any set dense in $\mathbb{R}$.)

Theorem 2.14 guarantees that this is an at most countable set. Observe that our set is really a subset of all finite subsets of $\mathbb{Q}$, which itself is a countable set.

Observe that this set is dense in $2^{\mathbb{R}}$. Consider an open set

$$U = \{f \in 2^{\mathbb{R}} : f(a_1) = \delta_1, \dots, f(a_n) = \delta_n\}$$

where $\delta_1, \dots, \delta_n \in \{0,1\}$. Then the point $f \in 2^{\mathbb{R}}$ such that $f(a_i) = \delta_i, f(x) = 0$ otherwise, is a point in U. Call this point y.

Since $\mathbb{R}$ is normal, there exist disjoint closed neighborhoods $[p_i, q_i]$ such that $a_i \in [p_i, q_i]$ Then observe that the set

$$\bigcup_{i=1}^{n} \{f \in 2^{\mathbb{R}} : f(x) = \delta_i \text{ if } x \in [p_i, q_i], f(x) = 0 \text{ otherwise}\}$$

is (1) a subset of A and (2) contains y. Therefore, A and U have a nonempty intersetion, and since U was an arbitrary open set of $2^{\mathbb{R}}$, we see that A is dense in $2^{\mathbb{R}}$. Since it is also countable, we have that $2^{\mathbb{R}}$ is separable, as desired.

■

Theorem 6.9 Let X be a 2^{nd} countable space. Then X is separable.

Proof: Let p_i be some point of B_i, $i \in \mathbb{N}$, where B_i is a basic open set from our countable basis. Then for any open set V of X, we know that V will intersect $\{p_i\}_{i\in\mathbb{N}}$ since by definition V must contain some basic open set B_i for which $p_i \in B_i$. Thus by Exercise 6.1, $\{p_i\}$ is dense, and since it's countable we have that X is separable.

■

Exercise 6.10

1. The space $\mathbb{R}_{\text{std}}$ is 2^{nd} countable (and hence separable).
2. The space $\mathbb{R}_{\text{LL}}$ is separable but not 2^{nd} countable.
3. The space $\mathbb{H}_{\text{bub}}$ is seprarble but not 2^{nd} countable.

Solution:

1. Consider the open set (a, b). Then observe that

$$\left(\bigcup_{p \in \mathbb{Q}, a \leq p} (p, \infty) \right) \cap \left(\bigcup_{q \in \mathbb{Q}, q \leq b} (\infty, q) \right) = (a, b).$$

   Therefore, we can generate (a, b) by open sets with rational endpoints, which shows that $\mathbb{R}$ has a countable basis. Specifically, the family of open sets $\{(p, q) : p, q \in \mathbb{Q}\}$ forms a countable basis for $\mathbb{R}_{\text{std}}$, so that by definition $\mathbb{R}_{\text{std}}$ is 2^{nd} countable.

2. By Exercise 6.2, we found that $\mathbb{R}_{\text{std}}$ is separable since the rationals form a countable, dense subset in $\mathbb{R}_{\text{std}}$. Thus every set (a, b) contains a rational. However, $(a, b) \subset [a, b)$, which which means that every set $[a, b)$ must also intersect the rationals. By Exercise 6.1, we can then conclude that the rationals are dense in $\mathbb{R}_{\text{LL}}$, and since they are countable this implies that $\mathbb{R}_{\text{LL}}$ is separable.

3. Observe that the positive rationals $\mathbb{Q}^+$ form a dense set of $\mathbb{R}^+ \cup 0$. That is, any set with (a, b) with $a, b \geq 0$ must contain a rational. By Theorem 6.5, we can then conclude that $(\mathbb{Q}^+)^2$ is dense in $(\mathbb{R}^+)^2$, and $\mathbb{Q}^+)^2$ is clearly countable. Thus by definition $(\mathbb{R}^+)^2$ is separable.

   However, observe that this is not 2^{nd} countable. Observe that if we are to cover this space, we need to cover the x-axis. But every point on the x axis needs an individual sticky bubble to cover it, and since there are an uncountable number of such x, it would be impossible to cover all of them with a countable number of sticky bubbles. Therefore this space is not 2^{nd} countable.

□

Theorem 6.11 Every uncountable set in a 2^{nd} countable space has a limit point.

Proof: Suppose we have an uncountable set A in X, and for the sake of contradiction suppose that U has no limit points. Then every point of A is an isolated point, which means that there exists an open set U such that $U \cap A = \{p\}$ for all $p \in A$. Note that for every such U there exists a B basic open set such that $B \subset U$. Thus $p \in B \subset U$. However, there are only countably many basic open sets, while an uncountable number of $p \in A$, which is a contradiction since we cannot contain an uncountable number of points with a countable number of basic open sets. Thus A must have a limit point, which is what we set out to show.

■

Theorem 6.14 Let X be 2^{nd} countable. Then X is 1^{st} countable.

Proof: Let X be a 2^{nd} countable space and $x \in X$. Then X has a countable basis $\mathcal{B}$. Consider the set $\mathcal{B}_x$ of all $B \in \mathcal{B}$ such that $x \in B$. Since $\mathcal{B}$ is a basis, we know that for any open set U containing x there exists a $B_x \in \mathcal{B}$ such that

$$p \in B_x \subset U.$$

But $p \in B_x$ so $B_x \in \mathcal{B}_x$. Therefore, $\mathcal{B}_x$ is a neighborhood basis of x. However, $\mathcal{B}_x \subset \mathcal{B}$ so $\mathcal{B}_x$ is countable. Therefore, every point of x as countable neighborhood basis so it is a 1^{st} countable space.

■

Theorem 6.15 If X is a topological space, $p \in X$, and p has a countable neighborhood basis, then p has a nested countable neighrborhood basis.

Proof: Let $\mathcal{B}$ be the countable neighborhood basis for p. Observe that for $B \in \mathcal{B}$, (1) $p \in B$ and (2) B is an open set, so by Theorem 3.3 we must be able to contain p in some neighborhood U such that $p \in U \subset B$. By the defintion of a neighborhood basis, there must exist another $B' \in \mathcal{B}$ such that $p \in B' \subset U$. Hence, for every $B \in \mathcal{B}$, there exists an element $B' \in \mathcal{B}$ such that

$$p \in B' \subset B.$$

Since $\mathcal{B}$ is countable, we can construct an at most countable set of nested open sets which form a neighborhood basis of p. Thus p has a nested countable neighborhood basis as desired.

■

Theorem 6.18 Suppose x is a limit point of the set A in a 1^{st} countable space X. Then there is a sequence of points $\{a_i\}_{i \in \mathbb{N}}$ in A that converges to x.

Proof: Since x is a limit point of A, for every open set U such that $x \in U$ we have that $(U - \{x\}) \cap A \neq \emptyset$. Since x is also a point in a first countable space, it has a countable neighborhood basis. By Theorem 6.15, x must therefore also have a nested countable neighborhood basis $\mathcal{B}$.

Since $\mathcal{B}$ is countable, we can write $\mathcal{B} = \{B_1, B_2, \ldots\}$. Let $i \in \mathbb{N}$. Now since each $B_i \in \mathcal{B}$ must contain some point $a_i \in A$, $a_i \neq x$, any open set of x will contain some a_i such that $a_i \in B_i \subset U$. Thus we must have that $\{a_i\}_{i \in \mathbb{N}}$ to be a sequence of points of A which converges to x.

■

Chapter 7

Compactness: The Next Best Thing To Being Finite

Theorem 7.1 Let X be a finite topological space. Then X is compact.

Proof: Consider an open cover $\mathcal{C}$ of the set X. Since $\mathcal{C}$ covers X, we know that for each $p \in X$ there exists an open set $U_p \in \mathcal{C}$ such that $p \in U_p$. Since X is finite, there are finitely many open sets $U_p \in \mathcal{C}$ such that $p \in U_p$. Therefore, we see that $\{U_p : p \in X\}$ is a finite subcover of $\mathcal{C}$, which shows that X is compact.

$\blacksquare$

Theorem 7.2 Let C be a compact subset of $\mathbb{R}_{\text{std}}$. Then C has a maximum point, that is, there is a point $m \in C$ such that for every $x \in C$, $x \leq m$.

Proof: Let $\mathcal{C}$ be an open cover of the set. Then it must have some finite subcover $\mathcal{C}'$. However, since the basic open sets of $\mathbb{R}$ are balls, there must be a finite set of basic open sets which cover C. However, every open set is of the form $(x - \epsilon, x + \epsilon)$, where $x \in C$ and $\epsilon > 0$. Take max x which appears in this finite open cover, and observe that for all $c \in C$, $c \leq x$. Thus C must have a maximum, as desired.

$\blacksquare$

Theorem 7.3 If X is a compact space, then every infinite subset of X has a limit point.

Proof: Consider an infinite subset A of X. Suppose that A has no limit points. Then for every $p \in X$, there exists an open set U_p such that $(U_p - \{p\}) \cap A = \emptyset$. However, this would imply that $\bigcup_{a \in A} U_a \subset X$, so that any open cover would automatically have to be infinite. But this contradicts the fact that X is compact. Therefore, A must have a limit point in X.

$\blacksquare$

Corollay 7.4 If X is compact and E is a subset of X with no limit point, then E is finite.

Proof: Suppose X is compact and a subset E has no limit point in X. Then for every $p \in X$, we know that there exists an open set U_p which contain p, $(U_p - \{p\}) \cap E = \emptyset$. Then again, $\bigcup_{p \in E} U_p \subset X$. Since every open cover of X must have a finite subcover, we know that there cannot be an infinite number of open sets U_p for $p \in E$, since we could then never cover it with finitely many open sets. Thus $\{U_p : p \in E\}$ has to be restricted to be finite, so that E must be finite, as desired.

■

Theorem 7.5 A space X is compact if and only if every collection of closed sets with the finite intersection property has a non-empty intersection.

Proof: Suppose X is compact, and let $\mathcal{C}$ be a collection of closed sets with the finite interscetion property. Suppose that $\bigcap_{C \in \mathcal{C}} C \neq \{p\}$ for some $p \in X$; this is a trivial case of the theorem.

Now let $p_1 \in C_1 \cap C_2$ for $C_1 \neq C_2$ and $C_1, C_2 \in \mathcal{C}$. We can then construct a sequence of points p_i such that

$$p_i \in C_1 \cap C_2 \cap \cdots \cap C_i \cap C_{i+1}$$

where $C_i, C_{i+1} \in \mathcal{C}$ and $C_i \neq C_{i+1}$ for all $i \in \mathbb{N}$. Since $\mathcal{C}$ has the finite intersection property, we know for a fact that we can always find a p_i in the finite intersection.

now if $\mathbb{C}$ has an empty intersection, then this implies that this sequence of points $\{p_i : i \in \mathbb{N}\}$ converges to a point p which is not contained in any $C \in \mathbb{C}$. First of all, we know it will converge to some point in X by Theorem 7.3. Second of all, observe that if p is the limit of this sequence, then for every open set U which contains p, there exists a $N \in \mathbb{N}$ such that for $i > N$, $p_i \in U$. Thus, in other words, if U contains p, then

$$(U - \{p\}) \cap C_i \neq \emptyset$$

for $i \in \mathbb{N}$. Thus p is a limit point for each C_i, and since each C_i is closed, $p \in C_i$ for all $i \in \mathbb{N}$.

Second attempt:

First we'll prove the forward direction. Suppose that X is a compact space, and let $\mathcal{C}$ be a collection of closed sets in X with the finite intersection property. For the sake of contradiction, suppose that $\bigcap_{C \in \mathcal{C}} C = \emptyset$. Then observe that $\{C^c : C \in \mathcal{C}\}$ (where ^{c} denotes the complement) is an open cover of X. Since this set is an open cover, it must have a finite subcover, which means that there exist sets $C_1^c, C_2^c \dots C_n^c$ such that

$$\bigcup_{i=1}^{n} C_i^c = X.$$

However, taking the complement of this leads to

$$\bigcap_{i=1}^{n} C_i = \emptyset$$

which contradicts the finite interscetion propety of $\mathcal{C}$. Thus we have a contradiction, which implies that there must exist a $\bigcap_{C\in\mathcal{C}} C \neq \emptyset$ as desired.

Now we prove the other direction. Suppose that for every collection $\mathcal{C}$ of closed sets in X with the finite interesection property, we have that

$$\bigcap_{C\in\mathcal{C}} C \neq \emptyset.$$

Now let $\mathcal{U}$ be an open cover of X. Suppose for the sake of contradiction that this does not have a finite subcover. Observe that the set $\{U^c : U \in \mathcal{U}\}$ is a collection of closed subsets in X with the finite intersection property, since each U^c is a closed set which is not disjoint with any other set. Since we know that every collection of closed sets in X with the finite interscetion property has a nonempty interesection, we can conclude that

$$\bigcap_{U\in\mathcal{U}} U^c \neq \emptyset \implies \bigcup_{U\in\mathcal{U}} U \neq X.$$

However, the last equation contradicts the fact that U was an open cover of X. Thus we have that every open cover must have a finite subcover, proving that X is compact as desired, completing the proof.

■

Theorem 7.6 A space X is compact if and only if for any open set U in X and any collection of closed sets $\{K_\alpha\}_{\alpha\in\lambda}$ such that $\cap_{\alpha\in\lambda} K_\alpha \subset U$, there exist a finite number of K_α's whose interesection lies in U.

Proof: First we'll prove the forward direction. Suppose X is compact and that $\cap_{\alpha\in\lambda_1} K_\alpha \subset U$ for some index λ_1. Observe that

$$U^c \subset \bigcup_{\alpha\in\lambda} K_\alpha^c$$

so that

$$\bigcup_{\alpha\in\lambda} K^c \cup U$$

is an open cover of X. Observe that this must have a finite subcover, so that λ_1 can at least be finite. Thus there can be a finite number of K_α's, given by $\{K_1, K_2, \dots, K_n\}$ such that

$$\bigcap_{i=1}^{n} K_i \subset U$$

which proves this direction.

Now we'll prove the other direction. Suppose that for every $U \subset X$ and any collection of closed sets $\{K_\alpha\}_{\alpha\in\lambda}$ such that $\bigcap_{\alpha\in\lambda} K_\alpha \subset U$, there exist a finite number of K_α's such that their intersection lies in U.
Now suppose that $\mathcal{U} = \{U_\alpha : \alpha \in \lambda\}$ is an open cover of X. Then observe that the set $\{U_\alpha^c\}_{\alpha\in\lambda}$ is a collection of closed sets such that $\bigcap_{\alpha\in\lambda} U_\alpha^c \subset \emptyset$. By assumption, there must exist a finite number of U_α^c's such that their intersection lies in $\emptyset$. Call these U_α's $U_1^c, U_2^c, \dots, U_n^c$. Then

$$\bigcap_{i=1}^{n} U_i^c \subset \emptyset \implies \bigcap_{i=1}^{n} U_i^c = \emptyset \implies \bigcup_{i=1}^{n} U_i = X$$

which shows that $\mathcal{U}$ must always have a finite subcover. Therefore, the space is compact as desired.

■

Exercise 7.7 If A and B are compact subsets of X, then $A \cup B$ is compact. Suggest and prove a generalization.

Solution: Suppose $\mathcal{W}$ is an open cover of $A \cup B$. Then observe that $\mathcal{W}$ is a cover of both A and B, and since A and B are compact, there exist finite subcovers of $\mathcal{W}$, denoted $\mathcal{W}_A$ and $\mathcal{W}_B$, such that $A \subset \mathcal{W}_A$ and $B \subset \mathcal{W}_B$. Now observe that $\mathcal{W}_A \cup \mathcal{W}_B$ is a finite subcover of $\mathcal{W}$, so that every open cover of $A \cup B$ has a finite subcover. Therefore $A \cup B$ is compact, as desired.

This can be extended to finitely many unions of compact sets. Suppose that $A_1, A_2, \dots, A_n$ are compact. Then $A_1 \cup A_2 \cup \dots A_n$ is compact. This is because any open cover $\mathcal{W}$ of $A_1 \cup A_2 \cup \dots A_n$ is also an open cover for each $A_1, \dots, A_n$, so there are finite subcovers $\mathcal{W}_{A_i}$ such that $\mathcal{W}_{A_i}$ covers A_i for $i = 1, 2, \dots, n$. Therefore, $\mathcal{W}_{A_1} \cup \dots \cup \mathcal{W}_{A_n}$ is a finite subcover of $\mathcal{W}$ containing $A_1 \cup \dots \cup A_n$, so that $A_1 \cup \dots \cup A_n$ is compact. However, this cannot be extended to infinitely many unions of compact sets since unioning infinitely many finite subcovers will not yield a finite subcover.

□

Theorem 7.8 Let A be a closed subspace of a compact space. Then A is compact.

Proof: Let X be compact and A a closed subspace of X. Then any closed set in A can be expressed as $D \cap A$, where D is closed in X. Since X is comapct, by Theorem 7.5, any collection of closed sets in X with the finite intersection property has a nonempty intersection. But closed sets in A are closed sets in X, so that any collection of closed sets in A with the finite intersection property have a nonempty interesection, which proves that A is a compact set. Therefore, A is compact.

■

Theorem 7.9 Let A be a compact subspace of a Hausdorff space X. Then A is closed.

Proof: Let $q \in X - A$. Since X is Hausdorff, for any $p \in A$, there exist disjoint open sets U_p and V_p such that $p \in U_p$ and $q \in V_q$. Now observe that the set $\{U_p | p \in A\}$ is an open cover of A, where each member corresponds to a disjoint open set V_p of the point q. Since A is a compact set, we know that the set must have a finite subcover; denote it as $\{U_{p_1}, U_{p_2}, \ldots, U_{p_n}\}$. Then the set $\bigcap_{i=1}^{n} V_{p_i}$ is a an open set containing q, (open because the interesection is finite) which is disjoint from A. Since this must hold for all $q \in X - A$, this shows that $X - A$ is an open set. Therefore, A is closed, as desired.

■

Exercise 7.10 Construct an example of a compact subset of a topological space that is not closed.

Solution: On the discrete topology, an finite set is an open set, although as we saw from Theorem 7.1 any finite set is also a compact set.

□

Exercise 7.11 Must the intersection of two compact sets be compact? Add hypothesis, if necessary. Extend any theorems you discover, if possible.

Theorem 7.12 Every compact, Hausdorff space is normal.

Proof: First we can show that X is regular. Suppose A is closed and consider any $p \notin A$. Then observe that, since X is Hausdorff, for each $a \in A$, there are disjoint open sets U_a and V_a such that $a \in U_a$ and $p \in U_a$. Then

$$U = \{U_a : a \in A\}$$

is an open cover of A, and since A is closed Theorem 7.8 guarantees that A is compact, and therefore there is a finite subcover

$$U' = \{U_a : a \in F\}$$

where F is a finite subset of A. Therefore, the set $V = \bigcap_{a \in F} V_a$ is an open set containing p but is entirely disjoint from all sets in U' by construction. Since A and $p \notin A$ were arbitrary, and we contained them in disjoint open sets, then we have that X is regular.

Now let A be closed and U be an open set containing A. Then note that for each $a \in A$ that $a \in B_a \subset U$ where B_a is some basic open set. Thus $\{B_a : a \in A\}$ is an open cover of A. By compactness of A, there must exist a finite subcover, given by $\{B_a : a \in F\}$ where F is a finite subset of A.

By regularity, we know that for each $a \in B_a$ there exists an open set V_a such that $a \in V_a$ and $\overline{V_a} \subset B_a$. Therefore, we see that $V = \bigcap\limits_{a \in F} V_a$ is an open set containing A and

$$\overline{V} \subset \bigcap_{a \in F} \overline{V_a} \subset U.$$

Thus we have contained A in an open set V such that $A \subset V$ and $\overline{V} \subset A$. By Theorem 5.9, we can conclude that X is normal, as desired.

■

Theorem 7.13 Let $\mathcal{B}$ be a basis for a space X. Then X is compact if and only if every cover of X by basic open sets in $\mathcal{B}$ has a finite subcover.

Proof: Suppose that X is compact and has a basis $\mathcal{B}$. Suppose that we cover X by basic open sets $B_{\alpha \in \lambda}$ such that $B_\alpha \in \mathcal{B}$ for all $\alpha \in \lambda$. Then because X is compact, there exsits a finite subcover, which we can express as $\{B_\alpha : \alpha \in \lambda'\}$ where λ' is a countable index. Thus we see that every cover of X by basic open sets in B has a finite subcover.

Now we prove the other direction. Suppose that every cover of X by basic open sets in $\mathcal{B}$ has a finite subcover. First observe that for any open cover $\mathcal{U} = \{U_\alpha : \alpha \in \lambda\}$, each

U_α can be expressed as the union of basis elements $\{B_{\gamma(\alpha)} : \gamma \in \lambda'\}$. If $\mathcal{U}$ covers X, then the set of basic elements $\{B_\gamma : \gamma \in \lambda\}$ will still contain X. But since every cover of X by basic open sets in the basis have a finite subcover, there exists a finite set which covers X which we can denote as $\{B_\alpha : \alpha \in \lambda''\}$, where λ'' is a finite index. Hence $\mathcal{U}$ has a finite subcover, which implies that X is a compact space.

■

Theorem 7.18 (The tube lemma) Let $X \times Y$ be a product space with Y compact. If U is an open set of $X \times Y$ containing the set $x_0 \times Y$, then there is some open set W in X containing x_0 such that U contains $W \times Y$ (called a "tube" around $x_0 \times Y$).

Proof: Let U be an open set in $X \times Y$ containing $x_0 \times Y$. Suppose for each $y \in Y$ we contain y in a set U_y and consider the product $U_x(y) \times U_y$, where $U_x(y) \times U_y \subset U$. Then since Y is compact, there exists a finite subcover of $\{U_y | y \in Y\}$. Suppose this is given by $\{U_{y_1}, \dots, U_{y_n}\}$. Then observe that

$$x_0 \subset \left(\bigcap_{i=1}^n U_{x(y_i)}\right) \times \left(\bigcup_{i=1}^n U_{y_n}\right) = \left(\bigcap_{i=1}^n U_{x(y_i)}\right) \times Y \subset U \times Y$$

so that $W = \bigcap_{i=1}^n U_{x(y_i)}$ is an open set in X such that $W \times Y \subset U$, as desired.

■

Theorem 7.19 Let X and Y be compact spaces. Then $X \times Y$ is compact.

Proof:

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Heine-Borel Theorem 7.20 Let A be a subset of $\mathbb{R}^n$ with the standard topology. Then A is compact if and only if A is closed and bounded.

Proof: Let $A \subset \mathbb{R}$ and suppose A is compact. Since $A \subset \mathbb{R}^n$, we now that it must be the product of compact sets $A_i \in \mathbb{R}$, $i = 1, 2, \dots n$. By Theorem 7.15, each such A_i must be closed and bounded. Hence their product, A, must also be closed and bounded, which proves this direction.

Now suppose A is closed and bounded. Then A must be a product of closed, bounded sets $[a_i, b_i]$ where $a_i \leq b_i$ and $i = 1, 2, \dots, n$. However, by Theorem 7.14, each such set is compact, and by Theorem 7.19 their product must also be compact. Hence, A is compact, which proves the theorem.

■

Alexander Subbasis Theorem 7.21 Let $\mathcal{S}$ be a subbasis for a space X. Then X is cmpact if and only if every subbasic open cover has a finite subcover.

Proof:

■

Tychonoff's Theorem 7.22 Any product of compacts sets is compact.

Proof:

■

Chapter 8
Continuity: When Nearby Points Stay Together

Theorem 8.1 Let X and Y be topological spaces and let $f : X \to Y$ be a function. Then the following are equivalent:

(a) The function f is continuous.

(b) For every closed set K in Y, the inverse image $f^{-1}(K)$ is closed in X.

(c) For every limit point p of a set A in X, the image $f(p)$ belongs to $\overline{f(A)}$.

(d) For every $x \in X$ and open set V containing $f(x)$, there is an open set U containing x such that $f(U) \subset V$.

Proof:

- $(1 \iff 2)$. Observe firstly that we can write $f^{-1}(K) = X - f^{-1}(Y - K)$. This is because every $x \in f^{-1}(Y - K)$ will be mapped to $Y - K$. Hence every $x \in X - f^{-1}(Y - K)$ will be mapped to K. Now observe that since K is closed (open), $Y - K$ is open (closed), so that $f^{-1}(Y - K)$ is an open (a closed) set in X. Therefore, we see that $X - f^{-1}(Y - K) = f^{-1}(K)$ is a closed (open) set, as desired.

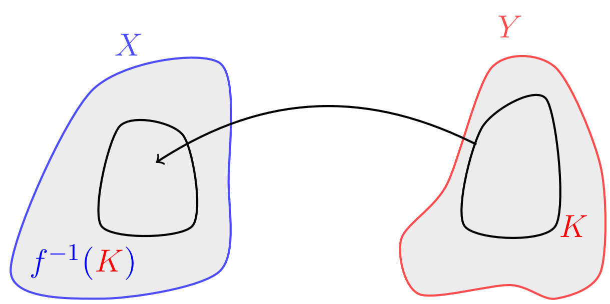


Figure 1: Note: f^{-1} may or may not be a function from $Y \to X$! This picture just demonstrates that for every closed $K \in Y$, $f^{-1}(K)$ is closed in X.

- $(1 \implies 3)$ Suppose for the sake of contradiction that $f(p)$ is an isolated point. Then there exists an open set U of Y such that $f(p) \in U$ but $U \cap f(A) = \emptyset$. Now observe that $f^{-1}(U)$ is open in X, and $p \in V$. But since p is a limit point of A, we know that $f^{-1}(U)$ must contain some $q \neq p$ and $q \in A$. However, this would imply that $f(q) \in U$, which is a contradiction since we assumed that $U \cap f(A) = \emptyset$. Thus we see that $f(p) \in \overline{f(A)}$.

- $(3 \implies 1)$ To prove the other direction, suppose that U is open in Y. Then let $U = Y - \overline{f(A)}$ for some set $A \in X$. Now by (3), we know that $f(\overline{A}) = \overline{f(A)}$. Hence

$$f^{-1}(U) = f^{-1}(Y - \overline{f(A)}) = f^{-1}(Y - f(\overline{A}))$$

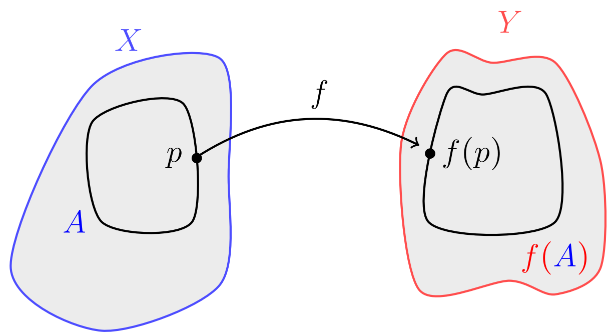


Figure 2: Here the limit point p of A maps to a limit point of $f(A)$.

maps to $X - \overline{A}$, which is an open set. Thus for every open set $U \subset Y$, we have that $f^{-1}(U)$ is open so that f is continuous.

- ($1 \implies 4$) Suppose that $f(x)$ is contained in some open $V \subset Y$ where $x \in X$. By definition of an open set, there must exist some U open in Y such that $f(x) \in U \subset V$. Now observe that, by continuity, $f^{-1}(U)$ is some open set in X such that $x \in f^{-1}(U)$ and $f(f^{-1}(U)) = U \subset V$. Thus there will always exist an open $W \in X$ such that $x \in W$ and $f(W) \subset V$, as desired.

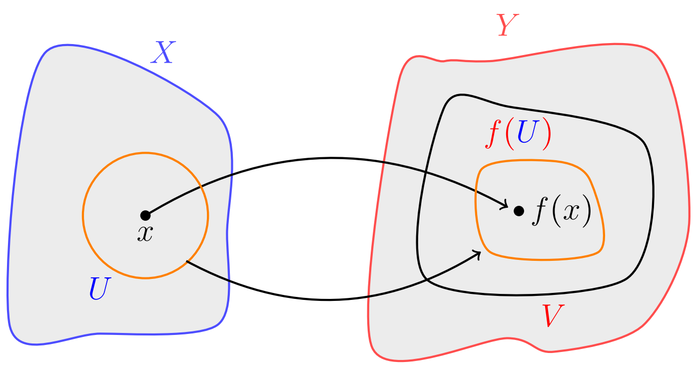


Figure 3: In this diagram we see that a neighborhood V of $f(x)$ corresponds to some open set U of x such that $f(U) \subset V$.

- ($4 \implies 1$) Consider an arbitrary open set V. By (4), we know that for each $f(x) \in V$, there exists a U open in X such that $f(U) \subset V$. Thus consider every $x \in X$ such that $f(x) \in V$, and let U_x be the open set in X such that $f(U_x) \subset V$. Then observe that

$$\bigcup_{x \in X \text{ s.t. } f(x) \in V} U_x = f^{-1}(V).$$

This is because for any $x \in \bigcup_{x \in X \text{ s.t. } f(x) \in V} U_x$, we know that $f(x) \notin Y - f(V)$ and for any $y \in f(V)$, there exists a $x \in \bigcup_{x \in X \text{ s.t. } f(x) \in V} U_x$ such that $f(x) = y$ by construction.

Since the arbitrary union of open sets is open, we thus see that $f^{-1}(V)$ is an open set. Since V was an arbitrary open set, this proves that f is continuous by definition.

■

Theorem 8.2 Let X, Y be topological spaces and $y_0 \in Y$. The constant map $f : X \to Y$ defined by $f(x) = y_0$ is continuous.

Proof: We can use the fourth property of continuity to prove this assertion. Observe that if V is any open set containing y_0, then $f^{-1}(V) = X$ because $f(X) = y_0$, as f is the constant map. If V is an open set not containing y_0, then $f^{-1}(V) = \emptyset$. Since X, $\emptyset$ are open, we have that the inverses of open sets in Y are open sets in X. Therefore, f is a continuous mapping.

■

Theorem 8.3 Let $X \subset Y$ be topological spaces. The inclusion map $i : X \to Y$ defined by $i(x) = x$ is continuous.

Proof: Observe that since if U is open in $X \subset Y$, then $i^{-1}(U) = U$ is open in X. Thus by the definition of continuity, we see that $i(x)$ is a continuous mapping.

■

Theorem 8.4 Let $f : X \to Y$ be a continuous map between topological spaces, and let A be a subset of X. Then the restriction map $f|_A : A \to Y$ defined by $f|_A(a) = f(a)$ is continuous.

Proof: Consider $a \in A \subset X$ and an open set V containing $f(a)$. Then there is an open set U of X containing a such that $f(U) \subset V$. Now observe that in the subspace topology, $A \cap U$ is an open set in A. Furthermore, $f(A \cap U) \subset V$, since $A \cap U \subset U$. Therefore, we see that for every $a \in A$ and open set V containing $f(a)$, there is an open set $A \cap U$ of A containing a such that $f(A \cap U) \subset V$, so that $f|_A : A \to Y$ is a continuous mapping.

■

Theorem 8.5 A function $f : \mathbb{R}_{\text{std}} \to \mathbb{R}_{\text{std}}$ is continuous if and only if for every $x \in \mathbb{R}$ and $\epsilon > 0$, there is a $\delta > 0$ such that for every $y \in \mathbb{R}$ with $d(x, y) < \delta$, then $d(f(x), f(y)) < \epsilon$.

Proof: First we'll prove the forward direction. Suppose that f is a continuous function from $\mathbb{R}_{\text{std}} \to \mathbb{R}_{\text{std}}$. By using the fourth property of continuous functions, we know that for every $x \in \mathbb{R}$ and open set V containing $f(x)$, there is an open set U containing x such that $f(U) \subset V$.

Since the basis for $\mathbb{R}_{\text{std}}$ consists of balls, we can let $V = B(f(x), \epsilon)$. Now we can take U to be a ball $B(x, \delta)$ such that for every $y \in B(x, \delta)$ we have that $f(y) \in B(f(x, \epsilon))$. In terms of the metric space, this means that the continuity of f implies that for every $\epsilon > 0$, there must be a $\delta > 0$ such that $d(x, y) < \delta \implies d(f(x), f(y)) < \epsilon$, which is the calculus definition of continuity.

Now we prove the other direction. Observe that if the calculus definition of continuity is given, then we can see that the freedom granted to $\epsilon > 0$ allows $d(f(x), y) < \epsilon$ to specify any arbitrary neighborhood V in $\mathbb{R}$ which contains $f(x)$, where $x \in \mathbb{R}$. Now the fact that we know there exists a $\delta > 0$ such that $d(x, y) < \delta \implies d(f(x), f(y)) < \epsilon$ implies the existence of an open set U containing x such that $f(U) \subset V$. This is exactly the fourth property of continuity offered in Theorem 8.1, which allows us to conclude that f is a continuous mapping, as desired.

■

***Q* (3/13/19)**: Is first countability necessary for the forward direction?

Theorem 8.6 Let X be a 1^{st} countable space and Y a topological space. Then a function $f : X \to Y$ is continuous if and only if for each convergent sequence $x_n \to x$ in X, $f(x_n)$ converges to $f(x)$ in Y.

Proof: First we prove the forward direction. Suppose that f is a continuous mapping, X is 1^{st} countable, and there is a sequence in X such that $x_n \to x$.

Consider an open set V in Y such that $f(x) \in V$. Since f is continuous, $U = f^{-1}(V)$ is an open set in X. Therefore there exists some $N \in \mathbb{N}$ such that for $i > N$, $x_i \in U$. Applying f to this last equation implies that $f(x_i) \in f(U) = f(f^{-1}(V)) = V$. Thus by definition, $f(x_i)$ is a sequence which converges to $f(x)$ in Y.

Next we prove the other direction. Suppose $x_n \to x$ and $f(x_n) \to f(x)$. For the sake of contradiction, suppose f is not continuous. By property (3) of Theorem 8.1, for some $A \subset X$ there exists a $p \in \overline{A}$ such that $f(p) \notin \overline{f(A)}$.

Since f is first countable, we know by Theorem 6.18 that there exists a sequence $\{a_i\}_{i \in \mathbb{N}}$ of points in A such that $a_i \to p$. Since a_i are points of A, we know that $f(a_i) \in f(A)$ for all $i \in \mathbb{N}$. However, we also know that $f(p) \notin \overline{f(A)}$, so that it could not be the case that $f(a_i) \to f(p)$. However, this is a contradiction, namely to our assumption. Therefore we must have that f is continuous.

■

Theorem 8.7 Let X by a space with a dense set D, and let Y be Hausdorff. Let $f : X \to Y$ and $g : X \to Y$ be continuous functions such that for every $d \in D$, $f(d) = g(d)$. Then for all $x \in X$, $f(x) = g(x)$.

Proof: Suppose that $f(x) \neq g(x)$ for some $x \notin D$. Since the points are distinct, and since Y is Hausdorff, there must exist disjoint open sets U, V in Y such that $f(x) \in U$ and $g(x) \in V$. Since both f, g are continuous, there must exist open sets U', V' in X such that $f(U') \subset U$ and $g(V') \subset V$. However, since D is dense in X, both U' and V' must intersect with some portion of D; that is, there is some $y \in U'$ and $z \in V'$ such that $y, z \in D$. Therefore, we see that $f(y) \in U$ and $g(z) \in V$. But we know that by definition of the function, $f(y) = g(z)$, which contradicts the fact that $U \cap V = \emptyset$. Therefore, we have a contradiction and it must be the case that $f(x) = g(x)$ for all $x \in X$, as desired.

■

Theorem 8.9 If $f : X \to Y$ and $g : Y \to Z$ are continuous then their composition $g \circ f : X \to Z$.

Proof: Consider an open set V in Z. Since g is continuous, we know that $g^{-1}(V)$ is open in Y. Since f is continuous, we also know that $f^{-1}(g^{-1}(V))$ is open in X. That is $(g \circ f)^{-1}(V)$ is open in X. Thus by definition $g \circ f$ is a continuous mapping.

■

Presented in class on 3/13/19

Theorem 8.10 (pasting lemma) Let $X = A \cup B$. where A, B are closed in X. Let $f : A \to Y$ and $g : B \to Y$ be continuous funtions that agree on $A \cap B$. Then the function $h : A \cup B \to Y$ such that $h = f$ on A and $h = g$ on B is continuous.

Proof: Consider K closed in Y. Then observe that $h^{-1}(K) = f^{-1}(K) \cup g^{-1}(K)$ is a union of closed sets in $A \cup B$. Thus the union is itself closed, so that for every K closed in Y we have that $h^{-1}(K)$ is closed in $A \cup B$, proving continuity.

■

Presented in class on 3/13/19

Theorem 8.11 (pasting lemma) Let $X = A \cup B$. where A, B are open in X. Let $f : A \to Y$ and $g : B \to Y$ be continuous funtions that agree on $A \cap B$. Then the function $h : A \cup B \to Y$ such that $h = f$ on A and $h = g$ on B is continuous.

Proof: Consider K open in Y. Then observe that $h^{-1}(K) = f^{-1}(K) \cup g^{-1}(K)$ is a union of open sets in $A \cup B$. Thus the union is itself open, so that for every K open in Y we have that $h^{-1}(K)$ is open in $A \cup B$, proving continuity.

■

Exercise 8.12 **Exercise 8.12** Is the pasting lemma true when A and B in the preceeding theorems are arbitrary sets?

Solution: The answer is no, since $A \cap B$ may turn out to be neither an open or closed set. Based on the definition $h : A \cup B \to Y$, it is possible that h would experience a discontinuity in transitioning from A to $A \cap B$ or B to $A \cap B$.

$\square$

Theorem 8.13 Let $f : X \to Y$ be a function and $\mathcal{B}$ a basis for Y. Then f is continuous if and only if for every open set B in $\mathcal{B}$, $f^{-1}(B)$ is open in X.

Proof: First we'll prove the forward direction. Suppose f is continuous. If $\mathcal{B} = \{B_\alpha : \alpha \in \lambda\}$ is a basis for Y, then observe that for every $\alpha \in \lambda$, $f^{-1}(B_\alpha)$ is an open set in X by the continuity of f, which proves this direction.

Next we prove the other direction. Suppose $\mathcal{B}$ is a basis for Y and for every $B \in \mathcal{B}$, $f^{-1}(B)$ is open in X. Then if V is open in Y, observe that

$$f^{-1}(V) = f^{-1}(\bigcup_{\alpha \in \lambda} B_\alpha) = \bigcup_{\alpha \in \lambda} f^{-1}(B_\alpha)$$

Since $f^{-1}(V)$ is a union of open sets in X, we have that $f^{-1}(V)$ is open in X. Since this holds for all V open in Y, we have that f is continuous.

$\blacksquare$

Theorem 8.14 Let $f : X \to Y$ be a function and $\mathcal{B}$ a subbasis for Y. Then f is continuous if and only if for every open set B in $\mathcal{B}$, $f^{-1}(B)$ is open in X.

Proof: First we'll prove the forward direction. Suppose f is continuous. If $\mathcal{B} = \{B_\alpha : \alpha \in \lambda\}$ is a basis for Y, then observe that for every $\alpha \in \lambda$, $f^{-1}(B_\alpha)$ is an open set in X by the continuity of f, which proves this direction.

Next we prove the other direction. Let $x \in X$ and suppose V is an open set in Y containing $f(x)$. Since $\mathcal{B}$ is a subbasis for Y, there must exists a finite set $\{B_i\}_{i=1}^n \subset \mathcal{B}$ such that $\bigcap_{i=1}^n B_i \subset V$. Therefore,

$$f^{-1}(V) \supset f^{-1}(\bigcap_{i=1}^n B_i) = \bigcap_{i=1}^n f^{-1}(B_i) = U.$$

where we have denoted $U = \bigcap_{i=1}^n f^{-1}(B_i)$. Since this is a finite intersection of open sets, each which contain x, we have that $U \subset f^{-1}(V)$. By property (3) of Theorem 8.1, we have that f is continuous as desired.

■

Theorem 8.15 If X is compact and $f : X \to Y$ is continuous and surjective, then Y is compact.

Proof: Consider an open cover $\mathcal{U} = \{U_\alpha : \alpha \in \lambda\}$ of Y. Since f is continuous, $f^{-1}(U_\alpha)$ is open in X for all $\alpha \in \lambda$. Since Y is closed in Y, $f^{-1}(Y)$ is a closed subspace in X. By theorem 7.8, $f^{-1}(Y)$ is therefore compact. Since $\{f^{-1}(U_\alpha) : \alpha \in \lambda\}$ is an open cover of $f^{-1}(Y)$ there exists a finite subcover, denoted as $\{f^{-1}(U_1), \dots, f^{-1}(U_n)\}$.

Since

$$f^{-1}(Y) \subset \bigcup_{i=1}^{n} f^{-1}(U_n) \implies Y \subset f(\bigcup_{i=1}^{n} f^{-1}(U_n)) = \bigcup_{i=1}^{n} f(f^{-1}(U_n)) = \bigcup_{i=1}^{n} U_n.$$

Thus we have that $\{U_1, \dots, U_n\}$ is a finite subcover of $\mathcal{U}$. Since $\mathcal{U}$ was an arbitrary open cover, this proves that Y is compact.

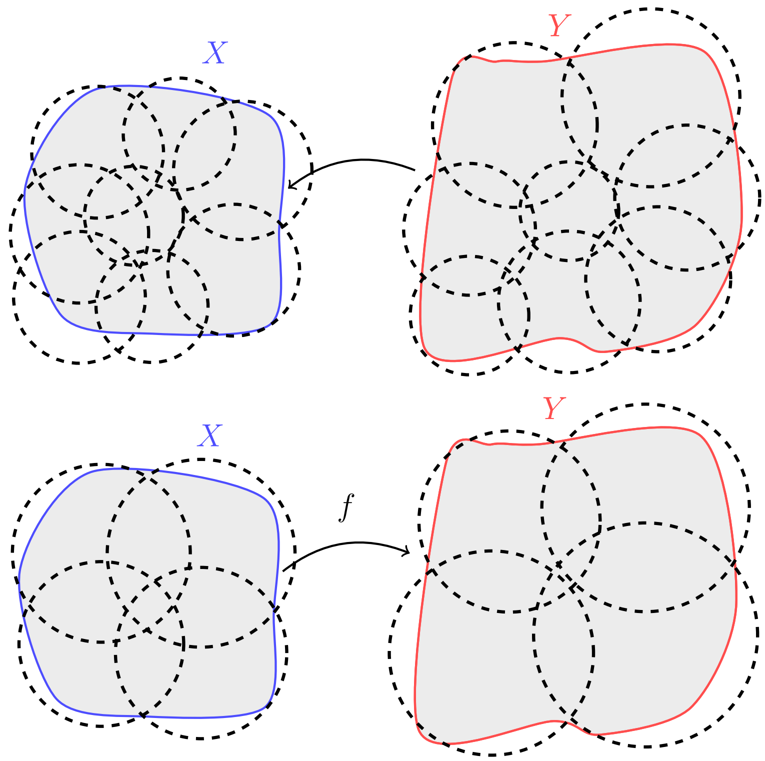


Figure 4: In the first diagram we start with an arbitrary cover of Y, and take the inverse open images in X. In the second diagram, we identify finite subcover, which exists since X is compact, and send this back into Y to obtain a finite subcover of Y.

■

Theorem 8.18 Let D be a dense subset of a topological space X and let $f : X \to Y$ be continuous and surjective. Then $f(D)$ is dense in Y.

Proof: Since f is surjective, we have that $f(X) = Y$. Now observe that

$$\overline{f(D)} = f(\overline{D}) = f(X) = Y$$

by property (3) of Theorem 8.1. Thus we have that $f(D)$ is dense in Y as desired.

■

Corollary 8.19 Let X be as separable space and let $f : X \to Y$ be continuous and surjective. Then Y is separable.

Proof: Since X is separable there exists a countable dense set A in X. Now observe that (1) $f(A)$ is at most countable since f is surjective and (2) $f(A)$ is also dense in Y by Theorem 8.18. Thus Y also has a countable dense subset so Y is separable.

■

Exercise 8.20

1. Find an open function that is not continuous.
2. Find a closed function that is not continuous.
3. Find a continuous function that is neither open nor closed.
4. Find a continuous function that is open but not closed.
5. Find a continuous function that is closed but not open.

Presented in class 3/25/19

Theorem 8.21 If X is normal and $f : X \to Y$ is continuous, surjective, and closed, then Y is normal.

Proof: Consider two disjoint closed sets A and B in Y. By continuity, $f^{-1}(A)$ and $f^{-1}(B)$ must be closed sets in X. As they are disjoint, and because X is normal, there must exist disjoint open sets U and V such that $f^{-1}(A) \subset U$ and $f^{-1}(B) \subset V$.

Observe that U^c, V^c are closed sets in X. By closedness of f, we know that $f(U^c), f(V^c)$ are closed sets in Y. Thus $f(U^c)^c = f(U)$ and $f(V^c)^c = f(V)$ are both open sets. Since $A \subset f(U)$ and $B \subset f(V)$, and $f(U)$ and $f(V)$ are disjoint as U, V are disjoint, we have that Y must be normal as desired.

■

Theorem 8.22 If $\{B_\alpha : \alpha \in \lambda\}$ is a basis for X and $f : X \to Y$ is continuous, surjective and open, then $\{f(B_\alpha)\}_{\alpha\in\lambda}$ is a basis for Y.

Proof: Suppose V is an open set in Y and $f(x) \subset V$ where $x \in X$. By continuity, there exists an open set $U \subset X$ such that $f(U) \subset V$. As $\{B_\alpha : \alpha \in \lambda\}$ is a basis for X, there exists a $B \in \{B_\alpha : \alpha \in \lambda\}$ such that $x \in B \subset U$ by Theorem 4.1. Therefore, $f(x) \in f(B) \subset V$.

Thus observe that

(a) $\{f(B_\alpha)_{\alpha\in\lambda}\} \subset \mathfrak{T}_Y$ since by openness of f each $f(B_\alpha)$ is open in Y for all $\alpha \in \lambda$ and

(b) for each open V in Y where $f(x) \in V$, there exists a $B \in \{B_\alpha : \alpha \in \lambda\}$ such that

$$f(x) \in f(B) \subset V.$$

As this satisfies Theorem 4.1, we have that $\{f(B_\alpha)\}_{\alpha\in\lambda}$ is a basis for Y.

■

Theorem 8.24 Let X be compact and Y be Hausdorff. Then any continuous function $f : X \to Y$ is closed.

Proof: Let A be a closed in X and consider $y \in Y - f(A)$. Let $\{U_\alpha\}_{\alpha\in\lambda}$ be an open cover of $f(A)$ where λ is an arbitrary index.

By continuity, we know that each $f^{-1}(U_\alpha)$ is open so

$$\{f^{-1}(U_\alpha)\}_{\alpha\in\lambda}$$

is an open cover of A in X. By Theorem 7.8, we know that A must be compact since it is a closed subspace of X, which is compact. Therefore, there exists a finite subcover of $\{f^{-1}(U_\alpha)\}_{\alpha\in\lambda}$, which we can denote as

$$\{f^{-1}(U_{\alpha_1}), \ldots, f^{-1}(U_{\alpha_n})\}.$$

Thus we know that $\{U_{\alpha_1}, \ldots, U_{\alpha_n}\}$. is an open cover of $f(A)$. Since every open cover of $f(A)$ has a finite subcover, we can conclude that $f(A)$ is compact. By Theorem 7.9, we have that $f(A)$ is closed since $f(A)$ is a compact subspace of Y which is a Hausdorff space. Therefore, f is a closed function.

■

Theorem 8.25 Being homeomorphic is an equivalence relation on topological spaces.

Proof: Let $X \sim Y$ if X is homeomorphic to Y.

Reflexive: Observe that $X \sim X$ since the identity function f is a continuous bijective function with a continuous inverse. Thus $\sim$ is reflexive.

Symmetric: If $X \sim Y$, there exsits a continuous bijective function $f : X \to Y$ with a continuous inverse. Observe that $f^{-1} : Y \to X$ is also a continuous bijective function with a continuous inverse, so that $X \sim Y \implies Y \sim X$. Thus $\sim$ is symmetric.

Transitive: If $X \sim Y$ and $Y \sim Z$, where $f : X \to Y$ and $g : Y \to Z$ are continuous bijective functions with continuous inverses, then obseve that $g \circ f$ is a continous bijective function (we proved earlier that compositions of continuous functions are continuious, and bijectivity is immediate) and that $(g \circ f)^{-1}$ is a also continuous since the inverses f^{-1} and g^{-1} are both continuous. Since $g \circ f : X \to Z$, we see that $X \sim Y$ and $Y \sim Z$ implies that $X \sim Z$. Thus $\sim$ is transitive

Since $\sim$ is reflexive, symmetric and transitive, we have that $\sim$ is an equivalence realtion as desired.

■

Presented in class on 3/25/19

Theorem 8.28 If $f : X \to Y$ is continuous, then the following are equivalent.

(a) f is a homeomorphism

(b) f is a closed bijection

(c) f is an open bijection.

Proof:

- (**a** $\implies$ **b**, **c**) Suppose f is a homeomorphism. Since f is continuous, we know for every closed (open) set $K \subset Y$ that $f^{-1}(K)$ is closed (open) in X. However, since f^{-1} is continuous, $f(f^{-1}(K)) = K$ is closed (open) in X. Since f is bijective, we thus have that every closed (open) set is uniquely mapped to another closed (open) set, and hence f is a closed (open) bijection.
- (**b**, **c** $\implies$ **a**) Suppose that f is a closed (open) bijection. Then every closed (open) set in X is mapped uniquely to a closed (open) set in Y by bijectivity. Thus f^{-1} is continuous. However, for every closed (open) set in K in Y, $f^{-1}(K)$ is closed (open) in X. Thus f is continous. Since f is continous, bijective and f^{-1} is continuous, we have that f is a homeomorphism.

■

Theorem 8.29 Suppose $f : X \to Y$ is a continuous bijection where X is compact and Y is Hausdorff. Then f is a homeomorphism.

Proof: Observe that we can apply Theorem 8.24 here, since f is continuous from X, a compact space, to Y, a Hausdorff space, to conclude that f is closed. Now since f is a continuous, closed bijection, we have by Theorem 8.28 that f is a homeomorphism, as desired.

■

Theorem 8.29 Let X be a compact space and let Y be Hausdorff. If $f : X \to Y$ is a continuous, injective map, then f is an embedding.

Proof: Observe it is given that f is injective while it is surjective, and hence bijective, into $f(X)$. Thus by Theorem 8.29 f is a homeomorphism from X to $f(X)$ since X is compact and $f(X)$ is Hausdorff. Therefore, f is an embedding.

■

Theorem 8.32 Let X and Y be topological spaces. The projection maps π_x, π_y, on $X \times Y$ are continuous, surjective, and open.

Proof: First observe that for every $x \in X$, $(x, y) \in X \times Y$ where $y \in Y$ so that $\pi_x(x, y) = x$. Therefore the function is surjective.

Now observe that it is continuous. Let U be open in X. Observe that

$$U \times Y \subset X \times Y \quad \text{and} \quad U \times Y = \pi_x^{-1}(U).$$

Since U, Y are open, $U \times Y$ is an open set in the product topology and hence $\pi_x^{-1}(U)$ is open. Thus π_x is continuous. Also, since $\pi_x(U \times Y) = U$ and U is open in X, we see that open sets in $X \times Y$ map to open sets in X, so that π_x is also an open function.

The proof that π_y is continuous, surjective and open is the exact same.

■

Q: Does the box topology behave similarly?

Theorem 8.33 Let X and Y be topological spaces. The product topology $X \times Y$ is the coarsest topology on $X \times Y$ that makes the projection maps π_x, π_y on $X \times Y$ continuous.

Proof: Observe that the product topology is generated by the subbasis of inverse images of open sets under the projection functions. Therefore, if we tried deleting any set from the product topology, there would exist an open set in X or Y such that its inverse image under the projection function is no longer open. Since we cannot remove any elements without making the projection functions discontinuous, we have that the product topology is the coarsest topology on $X \times Y$ that makes the projection functions continuus.

■

Presented on 3/27/19

Theorem 8.35 Let X and Y be topological spaces. For every $y \in Y$, the subspace $X \times \{y\}$ of $X \times Y$ is homeomorphic to X.

Proof: Consider the function $\pi'_x : X \times \{y\} \to X$, where $\pi'_x(x, y) = x$. I claim that this is a bijection, since for any $x_0 \in X$, $(x_0, y) \in X \times \{y\}$ so $\pi'_x(x_0, y) = x_0$. Thus the function is surjective. Now suppose $\pi'_x(x_1, y) = \pi'_x(x_2, y)$. Then $x_1 = x_2$, so the function is injective. Therefore it is bijective.

Let U be open in X. Then observe that (1) $U \times \{y\}$ is open in the subspace $X \times \{y\}$ and (2) $\pi'_x(U \times \{y\}) = U$. Therefore, π'_x is an open function.

Since π'_x is an open bijection, Theorem 8.28 guarantees that this is a homeomorphism. Therefore, we see that $X \times \{y\}$ is homeomorphic to X as desired.

■

Theorem 8.36 Let X, Y and Z be topological spaces. A function $g : Z \to X \times Y$ is continuous if and only if $\pi_x \circ g$ and $\pi_y \circ g$ are both continuous.

Proof: First we'll prove the forward direction. Suppose $g : Z \to X \times Y$ is continuous. Observe that $\pi_x \circ g$ and $\pi_y \circ g$ are both compositions of continuous functions (By Theorem 8.32, π_x and π_y are continuous functions) so $\pi_x \circ g$ and $\pi_y \circ g$ must be both continuous.

Now we prove the other direction. Suppose that $\pi_x \circ g$ and $\pi_y \circ g$ are both continuous. Consider an open set U in $X \times Y$. Since π_x and π_y are open functions by Theorem 8.32, we see that $\pi_x(U) = U_x$ is open in X and $\pi_y(U) = U_y$ is open in Y.

Now since $\pi_x \circ g : Z \to X$ and $\pi_y \circ g : Z \to Y$ are both continuous, $(\pi_x \circ g)^{-1}(U_x)$ and $(\pi_y \circ g)^{-1}(U_y)$ are both open functions in Z. Furthermore, if $U = U_x \times U_y$, then we can rewrite this as

$$U = \pi_x^{-1}(U_x) \cap \pi_y^{-1}(U_y) = (U_x \times Y) \cap (X \cap U_y).$$

Thus

$$\begin{aligned} g^{-1}(U) = g^{-1}(\pi_x^{-1}(U_x) \cap \pi_y^{-1}(U_y)) &= g^{-1}(\pi_x^{-1}(U_x)) \cap g^{-1}(\pi_y^{-1}(U_y)) \\ &= (\pi_x \circ g)^{-1}(U_x) \cap (\pi_y \circ g)^{-1}(U_y) \end{aligned}$$

which is the intersection of two open sets, by the continuity of $\pi_x \circ g$ and $\pi_y \circ g$. Since U was an arbitrary set and $g^{-1}(U)$ is open in Z, we have that g is a continuous function.
Use the fact that subbasis of inverse images of open sets in X and Y generate open sets in $X \times Y$.

■

Theorem 8.38 Let $\prod\limits_{\alpha\in\lambda} X_\alpha$ be the product of topological spaces $\{X_\alpha\}_{\alpha\in\lambda}$. The projection map $\pi_\beta : \prod\limits_{\alpha\in\lambda} X_\alpha \to X_\beta$ is a continuous, sujective, and open map.

Proof: Observe that we can show continuity as follows. If U is open in X_β, then observe that $f^{-1}(U) = \prod\limits_{\alpha\in\lambda\setminus\{\beta\}} \times U$ is a basic open set.

Next surjectivity follows from the fact that

$$\pi_\beta\left(\prod_{\alpha\in\lambda} X_\alpha\right) = X_\beta.$$

Finally, we see that for any basic open set, X_β can be restricted is an open set $U \subset X_\beta$. In this case, we see that $\pi_\beta\left(\prod\limits_{\alpha\in\lambda} X_\alpha\right) = U$ which is open in X_β. In the other case, X_β can be unrestricted, in which case $\pi_\beta\left(\prod\limits_{\alpha\in\lambda} X_\alpha\right) = X_\beta$. which is also an open set in X_β. Thus we see that π_β maps open sets in the product space to open sets in X_β, so it is an open function. Thus π_β is continuous, surjective and open.

■

Theorem 8.39 The product topology is the coarsest (smallest) topology on $\prod_{\alpha\in\lambda} X_\alpha$ that makes each projection map continious.

Proof: Observe that we can genereate the coarsest topology by generating a topology $\mathcal{T}$ such that $\pi_\beta^{-1}(U_\beta)$ is open for all $\beta \in \lambda$. Then finite intersections and arbitrary unions of these sets are open.

However, from Exercise 4.35 we found that the product topology is the topology generated by the subbasis of inverse images of open sets of the projection functions. Thus these topologies are equivalent, so that the product topology is the coarsest topology that keeps each projection map open.

■

Theorem 8.40 Let $\prod\limits_{\alpha\in\lambda} X_\alpha$ be the product of topological spaces $\{X_\alpha\}_{\alpha\in\lambda}$ and let Z be a topological space. A function $g : Z \to \prod\limits_{\alpha\in\lambda} X_\alpha$ is continuous if and only if $\pi_\beta \circ g$ is continuous for each $\beta \in \lambda$.

Proof: Suppose $g : Z \to \prod\limits_{\alpha\in\lambda} X_\alpha$ is continuous. Then observe that $\pi_\beta \circ g$ is a composition of continuous functions for all $\beta \in \lambda$, which proves this direction.

Now suppose that $\pi_\beta \circ g$ is continuous for all $\beta \in \lambda$. Let U be open in $\prod\limits_{\alpha\in\lambda} X_\alpha$. Observe that we can write U as

$$U = \bigcap_{i=1}^{n} \pi_{\beta_i}^{-1}(U_{\beta_i}) \quad U_\beta \in \mathfrak{T}_{X_{\beta_i}}, \beta_i \in \lambda.$$

Thus observe that

$$g^{-1}(U) = g^{-1}\left(\bigcap_{i=1}^{n} \pi_{\beta_i}^{-1}(U_{\beta_i})\right) = \bigcap_{i=1}^{n} g^{-1}(\pi_{\beta}^{-1}(U_{\beta_i})) = \bigcap_{i=1}^{n} (\pi \circ g)^{-1}(U_{\beta_i})$$

and since $\pi \circ \beta$ is continuous we know that $(\pi \circ g)^{-1}(U_{\beta_i})$ is open for all $\beta \in \lambda$. Therefore $\bigcap_{i=1}^{n}(\pi \circ g)^{-1}(U_{\beta_i})$ is a finite intersection of open sets, which is finite. Hence $g^{-1}(U)$ is open, proving that g is continuous, which completes the proof.

■

Theorem 8.41 Let $\mathbb{R}^\omega$ be the countably infinite product of $\mathbb{R}$ with itself. Let $\mathbb{R} \to \mathbb{R}^\omega$ be defined by $f(x) := (x, x, x, \dots)$. Then f is continuous if $\mathbb{R}^\omega$ is given the product topology, but not if given the box topology.

Proof: Consider $x \in \mathbb{R}$ and an open set V in $\mathbb{R}^\omega$ containing $f(x)$. Let B be the basic open set such that

$$f(x) \in B \subset V.$$

Now $B = \prod_{\alpha\in\lambda} U_\alpha$ where U_α are open in $\mathbb{R}$ and $U_\alpha = \mathbb{R}$ except for a finite number of $\alpha \in \lambda' \subset \lambda$. Since $f(x) = (x, x, x, \dots) \in B$, we know that

$$x \in U_\alpha, \quad \alpha \in \lambda'.$$

The fact that $x \in U_\alpha$ where $\alpha \in \lambda - \lambda'$ is obvious, since $U_\alpha = \mathbb{R}$ for $\alpha \in \lambda - \lambda'$. Therefore,

$$x \in \bigcap_{\alpha\in\lambda'} U_\alpha$$

and because this is a finite interection, $U = \bigcap\limits_{\alpha\in\lambda'} U_\alpha$ is open in $\mathbb{R}$. Since

$$x \in U \qquad f(U) \subset B \subset V,$$

we have that $f : \mathbb{R} \to \mathbb{R}^\omega$ is a continuous function as x was an arbitrary member of $\mathbb{R}$.

Next suppose $\mathbb{R}^\omega$ is endowed with the box topology. Construct an open set V in $\mathbb{R}^\omega$ containing $f(x) = (x, x, x, \dots)$ as follows. Let

$$V_n = \left(x - \frac{1}{n}, x + \frac{1}{n}\right) \quad n \in \mathbb{N}$$

so that $V = \prod_{n\in\mathbb{N}} V_n$. Observe that (1) $f(x) \in V$ and (2) there is no open set $U \subset \mathbb{R}$ such that $f(U) \subset V$ because

$$\bigcap_{n\in\mathbb{N}} V_n = \bigcap_{n\in\mathbb{N}} \left(x - \frac{1}{n}, x + \frac{1}{n}\right) = \{x\}.$$

Therefore, we see that f is not continuous under the box topology.
Q: What functions are continuous under the box topology, but *not* the product topology? Or are all functions which are continuous under the box topology continuous for the product topology? Also, what happens when ω becomes uncountable? I think it would still work...

■

Theorem 8.42 The cantor set is homeomorphic to the product $\prod_{n\in\mathbb{N}}\{0,1\}$ where $\{0,1\}$ has the discrete topology.

Proof: Consider $c \in \mathcal{C}$, the Cantor set, and let C_n be the cantor ternary sets such that

$$C_n = \frac{C_{n-1}}{3} + \left(\frac{2}{3} + \frac{C_{n-1}}{3}\right) \quad C_0 = [0,1].$$

and $\mathcal{C} = \bigcap_{n=1} C_n$. Let us define $f : \mathcal{C} \to \prod_{n\in\mathbb{N}}\{0,1\}$ as follows:

$$f(c) = (i_1(c), i_2(c), \dots)$$

where

$$i_n(c) = \begin{cases} 1 & \text{if } c \in C_n \\ 0 & \text{if } c \in \left(\frac{2}{3} + \frac{C_n}{3}\right) \end{cases} \quad n \geq 1.$$

Now we will demonstrate continuity. Let U be a basis element of $\prod_{n\in\mathbb{N}} = \{0,1\}$. Then $U = \prod_{\alpha\in\lambda} U_\alpha\{0,1\}$ and $U = \{0,1\}$ for all but a finite number of $\alpha \in \lambda' \subset \lambda$. For each $\alpha \in \lambda'$, $U_\alpha = i(\alpha)$ where $i(\alpha) = 1$ or 0. Thus consider the set of points

$$\{(\dots, i(\alpha_1), \dots, i(\alpha_2), \dots) : \alpha_i \in \lambda'\}$$

which are simply a subset of the cantor set which are restricted to finitely many C_n's.

This is surjective since for any $p \in \prod_{n\in\mathbb{N}}\{0,1\}$, we can find a point $c \in \mathcal{C}$ such that $f(c) = p$ as follows: if $p_i = 0$ or 1, then we place c in C_i or $\frac{2}{3} + \frac{C_i}{3}$. Continuing inductively, which we can since this is a countable process, we'll eventually generate a point $c \in \mathcal{C}$ for which $f(c) = p$.

Now observe that this is injective. Note that

$$f(c_1) = f(c_2) \implies (i_1(c_1), i_2(c_1), \dots) = (i_1(c_2), i_2(c_2), \dots)$$

Since every coordinate must be equal, we see that the location of the two points c_1 and c_2 must be equal; hence, $c_1 = c_2$.

Finally observe that f open, since the image of every mapping consist of fixating a finite number of 1's and hence corresponds to specifying a finite number of elements in the image. By definition, this is an open set in the product space.

Now since f is an open, continuous bijection, we have by Theorem 8.28 that f is a homeomorphism. Therefore, $\prod_{n\in\mathbb{N}}\{0,1\}$ is homeomorphic to the Cantor set.

■

Exercise 8.45 A torus is the surface of a doughnut. Construct a torus as

1. an identification space of C, the cylinder
2. an identification space of $X = [0,1] \times [0,1]$
3. an identification space of $\mathbb{R}^2$.

Solution:

1. To construct a torus from a cylinder, we simply glue the ends of the cylinder together in a continuous fashion.

2. To construct a from X, we identify points together the points from the top and bottom of the box and identify the points together on the sides of the box. Thus we can construct a torus as

$$T = \Big\{\{(x,0) \cup (x,1)\} : x \in (0,1)\Big\} \cup \Big\{\{(0,y) \cup (1,y)\} : y \in [0,1]\Big\}$$

using the identification space X.

3. With an identification space of $\mathbb{R}$, we see that can create the map

$$T = \Big\{\{(x,-\infty)\} \cup \{(x,\infty)x \in \mathbb{R}\}\Big\} \cup \Big\{\{(-\infty),y\} \cup \{(\infty,y)y \in \mathbb{R}\}\Big\}$$

□

Exercise 8.46 Describe the 2-dimensional sphere (the boundary of a 3 dimensional ball in $\mathbb{R}^3$) as an identification space of two discs in $\mathbb{R}^2$ by drawing a figure.

Solution:

Consider two discs of radius 2 centered at $x = (-1, 0)$ and $x = (1, 0)$. Then we can construct a sphere with the map

$$\Big\{\{(x, \sqrt{4-(x-2)^2}) \cup (-x, \sqrt{4-(x-2)^2}) : x \in [0,4]\}$$
$$\cup\Big\{\{(x, -\sqrt{4-(x+2)^2}) \cup (x, -\sqrt{4-(x+2)^2}) : x \in [-4,0]\}\Big\}$$

while the points in the interior of the disk get symmetrical mapped to the bottom and top hemispheres.

□

Theorem 8.47 The quotient topology actually defines a topology

Proof: We can verify the collection of sets $\mathcal{T}$ in Y actually forms a topology by verifying the four properties.

1. Observe that $\emptyset \subset Y$ and $f^{-1}(\emptyset) = \emptyset$, an open set in X, so that $\emptyset \in \mathcal{T}$.
2. Since f is surjective, we know that $f(X) = Y$. Hence $f^{-1}(Y) = X$, an open set, so that $Y \in \mathcal{T}$.
3. Observe that if $U, V \in \mathcal{T}$, then $f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$. As this is the intersection of two open sets in X, we see that $f^{-1}(U \cap V)$ is open in X so that $U \cap V \in \mathcal{T}$.
4. Finally, suppose $\{U_\alpha\}_{\alpha \in \lambda}$ is a collection of sets in $\mathcal{T}$. Then observe that

$$f^{-1}(\bigcup_{\alpha \in \lambda} U_\alpha) = \bigcup_{\alpha \in \lambda} f^{-1}(U_\alpha).$$

The right hand side is the arbitrary union of open sets in X, which is also open in X. Hence $f^{-1}(\bigcup_{\alpha \in \lambda} U_\alpha)$ is open in X so that $U_\alpha \in \mathcal{T}$ for all $\alpha \in \lambda$.

With the four properties proven, we can now conclude that this does form a topology.

■

Theorem 8.48 Let X be a topological space, Y be a set, and $f : X \to Y$ a surjective map. The quotient topology on Y is the finest (largest) topology that makes f continuous.

Proof: Suppose we try adding a set U to the quotient topology. Then $f^{-1}(U)$ is not open, since if it were then U would already be in the topology, and thus f would no longer be continuous. Thus it is the largest topology that makes f continuous.

■

Theorem 8.53 Let $f : X \to Y$ be a quotient map. Then a map $g : Y \to Z$ is continuous if and only if $g \circ f$ is continuous.

Proof: First we prove the forward direction. Suppose that $f : X \to Y$ is a quotient map and $g : Y \to Z$ is continuous. By Theorem 8.9, the composition $g \circ f$ must also be continuous, which proves this direction.

Now suppose that $g \circ f$ is continuous. If U is an open set in Z, then $(g \circ f)^{-1}(U)$ is open in X. Note that $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$. Since this is open in X, and because f is a quotient map, it must be that $g^{-1}(U)$ is open in Y. Therefore, we have that $g : Y \to Z$ is a continuous mapping, which proves this direction and completes the proof.

■

Exercise 8.54 Let the cylinders C^* and C be defined as at the beginning of this section. Prove that C^* is homeomorphic to C by constructing a map $h : C^* \to C$ and showing it is a continuous bijection from a compact space into a Hausdorff space.

Chapter 9
Connectedness: When Things Don't Fall Into Pieces

Theorem 9.1 The following are equivalent:

1. X is connected
2. there is no continuous function $f : X \to \mathbb{R}_{\text{std}}$ such that $f(X) = \{0, 1\}$
3. X is not the union of two disjoint nonempty separated sets
4. X is not the union of two disjoint nonempty closed sets
5. the only subsets of X that are both closed and open in X are both the empty set and X itself
6. for every pair of points p and q and every open cover $\{U_\alpha\}_{\alpha\in\lambda}$ of X there exists a finite number of U_α's, $\{U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n}\}$ such that $p \in U_{\alpha_1}$, $q \in U_{\alpha_n}$ for each $i < n$, $U_{\alpha_i} \cap U_{\alpha_{i+1}} \neq \emptyset$.

Proof:

(**1** $\implies$ **2**) Suppose X is connected, and for contradiction that there is a continuous function $f : X \to \mathbb{R}_{\text{std}}$ such that $f(X) = \{0, 1\}$. However, this would imply that $f^{-1}(1)$ and $f^{-1}(0)$ are (1) disjoint open sets in X such that (2) their union is X. However, that contradicts the fact that X is connected by definition. Therefore, there is no continuous function $f : X \to \mathbb{R}_{\text{std}}$ such that $f(X) = \{0, 1\}$.

(**2** $\implies$ **1**) Now if there is no continuous function $f : X \to \mathbb{R}_{\text{std}}$ such that $f(X) = \{0, 1\}$, then that means X cannot be split into two disjoint open sets whos union is X, which implies that X is connected.

(**1** $\implies$ **3**) Since X is connected, it is not the union of two nonempty disjoint open subsets of X. However, suppose A, B are two separated sets such that $A \cup B = X$.

(**3** $\implies$ **1**) Suppose now that X is not the union of two disjoint nonempty separated sets. Then X is not union of two disjoint open sets, so that X is connected.

(**1** $\implies$ **4**) Suppose X is connected, and for contradiction that $X = A \cup B$ where A and B are disjoint nonempty closed sets. Then we can construct a continuous function from $f : X \to \{0, 1\}$, where $f^{-1}(0) = A$ and $f^{-1}(1) = B$. However, this contradictions the fact that X is connected, so that X is no the union of two disjoint nonempty closed sets.

(**4** $\implies$ **1**) Suppose X is not the union of two disjoint nonempty closed sets. Then there is no continuous function $f : X \to \{0, 1\}$ since $f^{-1}(0)$ and $f^{-1}(1)$ cannot be open or closed. Thus X must be connected.

(**1** $\implies$ **5**) Suppose X is connected. Suppose there is a set such that $A \neq X$

and $A \neq \emptyset$ is open and closed. Then $A^c \cup A = X$. However, that would mean X is the union of two disjoint non empty open sets, which is a contradiction. Thus the only open and closed sets are X and $\emptyset$.

(**5** $\implies$ **1**) Suppose the only open and closed sets in X are X and $\emptyset$. Suppose for contradiction that X is not connected, so that $X = A \cup B$ for two disjoint nonempty open sets. Then $X^c = (A \cup B)^c = A^c \cap B^c = \emptyset$. However, this is a contradiction since their intersection must be nonempty. Therefore, X is connected.

$\blacksquare$

Exercise 9.2 **Exercise 9.2** Which of the following spaces are connected?

1. $\mathbb{R}$ with the discrete topology?
2. $\mathbb{R}$ with the indiscrete topology?
3. $\mathbb{R}$ with the finite complement topology?
4. $\mathbb{R}_{\text{LL}}$?
5. $\mathbb{Q}$ as a subspace of $\mathbb{R}_{\text{std}}$?
6. $\mathbb{R} - \mathbb{Q}$ as a subspace of $\mathbb{R}_{\text{std}}$?

Solution:

1. Every subset of $\mathbb{R}$ is open and closed. This violates Theorem 9.1(5) so that $\mathbb{R}$ is not connected under the discrete topology.

2. The only sets which are open and closed are $\mathbb{R}$ and $\emptyset$. Thus by Theorem 9.1(5) $\mathbb{R}$ is connected under the indiscrete topology.

3. For contradiction suppose there is a set $U \subset \mathbb{R}$ which is open and closed and not $\mathbb{R}$ or the emptyset.

   Since U is open, U^c is finite. However $(U^c)^c = U$ is infinite and hence U^c is not an open set. But this contradicts the assumption that U was open and closed. Thus $\mathbb{R}$ is connected on the finite complement topology.

4. Consider a basic open set $[a, b)$. Observe that

   $$[a, b)^c = (-\infty, a) \cup [b, \infty)$$

   which is the union of two open sets, and hence is open. Thus $[a, b)$ is open and closed. By Theorem 9.1.5, we have that $\mathbb{R}_{\text{LL}}$ is not connected.

5. Observe that $(\mathbb{Q} \cap (-\infty, \pi))$ and $(\mathbb{Q} \cap (\pi, \infty))$ are disjoint, separated sets in the subspace $\mathbb{Q}$ and

   $$(\mathbb{Q} \cap (-\infty, \pi)) \cap (\mathbb{Q} \cap (\pi, \infty)) = \mathbb{Q}.$$

   Thus $\mathbb{Q}$ is not open as a subspace of $\mathbb{R}_{\text{std}}$.

6. Observe that $(\mathbb{R}-\mathbb{Q})\cap(-\infty,0)$ and $(\mathbb{R}-\mathbb{Q})\cap(0,\infty)$ are disjoint separated sets and
$$((\mathbb{R}-\mathbb{Q})\cap(-\infty,0))\cup((\mathbb{R}-\mathbb{Q})\cap(0,\infty))=\mathbb{R}-\mathbb{Q}.$$
Thus $\mathbb{R}-\mathbb{Q}$ is not connected.

□

Theorem 9.3 The space $\mathbb{R}_{\text{std}}$ is connected.

Proof: The only closed and open sets in $\mathbb{R}_{\text{std}}$ are the emptyset and $\mathbb{R}$ itself, so that by Theorem 9.1(5) we can conclude that $\mathbb{R}_{\text{std}}$ is connected.

■

Theorem 9.4 Let A and B be separated subsets of a space X. If C is a connected subset of $A\cup B$, then either $C\subset A$ or $C\subset B$.

Proof: Observe that if C is a connected subset of $A\cup B$, where A and B are separated in X, then C is not the union of two disjoint open sets in the $A\cup B$ subspace topology.

Suppose for the sake of contradiction that $C\subset A$ and $C\subset B$. Then observe that
$$C\subset A\cap B=\emptyset$$
which is a contradiction since C is nonempty. Thus it must be that $C\subset A$ or $C\subset B$.

■

Theorem 9.5 Let $\{C_\alpha\}_{\alpha\in\lambda}$ be a collection of connected subsets of X and E another connected subset of X that for each $\alpha\in\lambda$, $E\cap C_\alpha\neq\emptyset$. Then $E\cup(\bigcup\limits_{\alpha\in\lambda} C_\alpha)$ is connected.

Proof: Suppose for the sake of contradicition that $E\cup(\bigcup\limits_{\alpha\in\lambda} C_\alpha)$ is not connected. Then $E\cup(\bigcup\limits_{\alpha\in\lambda} C_\alpha)=A\cup B$ where A and B are some separated sets in X. Observe that since E is a connected subset of X, we have by Theorem 9.4 that $E\subset A$ or $E\subset B$. Without loss of generality suppose $E\subset A$. Then since each C_α is a connected subset of $A\cup B$, Theorem 9.4 implies that $C_\alpha\subset B$ for at least one $\alpha\in\lambda$. However, this is a contradiction since $E\cap C_\alpha\neq\emptyset$ for all $\alpha\in\lambda$, while $A\cap B=\emptyset$. Therefore, we must have that $E\cup(\bigcup\limits_{\alpha\in\lambda})$ is connected.

■

Theorem 9.6 Let C be a connected subset of the topological space X. If D is a subset of X such that $C\subset D\subset\overline{C}$, then D is connected.

Proof: Suppose that C is a connected subset of X and for the sake of contradiction that D such that $C \subset D \subset \overline{C}$ is not connected. Then there exists disjoint open sets A and B such that $A \cup B = D$. Since C is connected, we know by Theorem 9.5 that $C \cap A = \emptyset$ or $C \cap B = \emptyset$. Without loss of generality, suppose that $C \cap A = \emptyset$. Then this is a contradiction since $A \subset D \subset \overline{C}$. Therefore, we must have that D is connected.

■

Theorem 9.8 Let X be a topological space, C a connected subset of X, and $X - C = A \big| B$. Then $A \cup C$ and $B \cup C$ are each connected

Proof: Suppose that $X - C = A \cup B$ where A and B are separated. Now suppose that $A \cup C$ is not connected, so that $A \cap C = U \cup V$ where U, V are open. Now suppose that $U \cap C \neq \emptyset$ and $V \cap C \neq \emptyset$. Then $(U \cap C) \cup (V \cap C) = A \cap C$

■

Presented in class 4/3/19

Theorem 9.12 Let $f : X \to Y$ be a continuous, surjective function. If X is connected, then Y is connected.

Proof: Suppose $f : X \to Y$ is a continuous, surjective function. We can do proof by contradiction. Suppose X is connected but Y is not connected. By Theorem 9.1 part 5, there exists a set $V \subset Y$, $V \neq \emptyset$ $V \neq Y$, such that V is open and closed in Y. By continuity, $f^{-1}(V)$ is both open and closed in X, and by surjectivity, $f^{-1}(V)$ is a proper subset of X. Thus X has an open and closed set, one which is not $\emptyset$ or X, which contradicts the fact that X is not connected by Theorem 9.1 part 5. Thus if X is connected, Y is connected, as desired.

■

Theorem 9.13 (Intermediate Value Theorem!) Let $f : \mathbb{R}_{\text{std}} \to \mathbb{R}_{\text{std}}$ be a continuous map. If $a, b \in \mathbb{R}$ and r is a point of $\mathbb{R}$ such that $f(a) < r < f(b)$ then there exists a point c in (a, b) such that $f(c) = r$

Proof: Observe that $\mathbb{R}_{\text{std}}$ is connected. Since $f : \mathbb{R}_{\text{std}} \to \mathbb{R}_{\text{std}}$, connected should be preserved.

Suppose there does not exist a point $c \in (a, b)$ such that $f(c) = r$. Then $f(x) < r$ or $r < f(x)$ for all $x \in (a, b)$. However since $f(\mathbb{R}) = \mathbb{R}$, this implies that $\mathbb{R}_{\text{std}}$ is not connected, which contradicts the fact that $\mathbb{R}_{\text{std}}$ is connected. Therefore such a c must exist.

■

Theorem 9.18 Each component of X is connected, closed, and not contained in any strictly larger connected subset of X.

Proof: Consider a component $C = \bigcup_{\alpha \in \lambda} C_\alpha$ of p in X, where each C_α is connected and $p \in C_\alpha$ for all $\alpha \in \lambda$. Observe that we can apply Theorem 9.5 to conclude that C is connected, since (1) no member of the union of C is disjoint from any other member (as they all contain p) and (2) each member is connected.

Suppose that C is not closed. Then there is a point $q \notin C$ and an open set U containing q such that $(U - \{q\}) \cap C \neq \emptyset$.

$\blacksquare$

Theorem 9.35 A path connected space is connected.

Proof: Suppose X is path connected but not connected. Then there exist two disjoint open subsets A, B such that $A \cup B = X$. Observe that any point in A cannot be joined together with any point B by a path, a contradiction to the path connectivity of X. Thus X must be connected.

$\blacksquare$

Theorem 9.36 The flea and comb space is connected but not pathwise connected. (The flea and comb space is the union of the topologist's comb and the point (0, 1).)

Proof: Let A be the set of the comb space. This is obviously path connected, and so it is connected by Theorem 9.35. Observe now that

$$A \subset A \cup flea \subset \overline{A}$$

so that $A \cup flea$, the flea and comb space, must be connected.

$\blacksquare$

Chapter 10
Metric Spaces: Getting some distance

4/8/19 Q: Why does X have to be a metric space?

Lebesgue Number Theorem 10.24 Let $\{U_\alpha\}_{\alpha\in\lambda}$ be an open cover of a compact set A in a metric space X. Then there exists a $\delta > 0$ such that for every point $p \in A$, $B(p,\delta) \subset U_\alpha$ for some α. This number δ is called a **Lebesgue number** of the cover.

Proof: Since A is compact, there exists a finite subcover $\{U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n}\}$. Now suppose for the sake of contradiction that there does not exists such a δ. Then in order for this to happen, we would need that for every $B(p,\delta)$ containing p there exists a member of $U_{\alpha'} \in \{U_{\alpha_1}, U_{\alpha_2}, \ldots, U_{\alpha_n}\}$ such that $B(p,\delta) \not\subset U_{\alpha'}$. But since we can theoretically propose an infinite number of δ, we must have an infinite number of such U'_αs.

However, we cannot do this in the finite subcover, as it is finite. Therefore the contrary must be true: there exists a δ such that for every $p \in A$, $B(p,\delta) \subset U_\alpha$ for some α. And since this is true for the finite subcover, which is a subset of the open cover, this is definitely true for the open cover.

■

Theorem 10.25 Let $\gamma : [0,1] \to X$ be a **path**: a continuous map from $[0,1]$ into the space X. Given an open cover $\{U_\alpha\}$ of X, show that $[0,1]$ can be divided into N intervals of the form $I_i = [\frac{i-1}{N}, \frac{i}{N}]$ such that each $\gamma(I_i)$ lies completely in one set of the cover.

Proof: If $\{U_\alpha\}_{\alpha\in\lambda}$ is an open cover of X, then consider the set $\{\gamma^{-1}(U_\alpha)\}_{\alpha\in\lambda}$. This will be an open cover of γ, since we know γ maps $[0,1]$ into X. However, since γ is compact, we know by Lebesgue Number Theorem that there exists a δ such that $p \in B(p,\delta) \subset \gamma^{-1}(U_\alpha)$ for all $p \in [0,1]$ where $\gamma^{-1}(U_\alpha)$ is some set in the open cover containing p.

Let $\dfrac{1}{N} < \delta$ where N is a positive integer. Then observe that the sequence of intervals

$$\left[\frac{i-1}{N}, \frac{i}{N}\right] \quad 1 \le i \le N$$

will each be contained in at least one member of $\gamma^{-1}(U_\alpha)$. Thus

$$\left[\frac{i-1}{N}, \frac{i}{N}\right] \subset \gamma^{-1}(U_\alpha) \implies \gamma\left(\left[\frac{i-1}{N}, \frac{i}{N}\right]\right) \subset U_\alpha.$$

Thus $[0,1]$ can be divided into N intervals of the form $I_i = [\frac{i-1}{N}, \frac{i}{N}]$ such that each $\gamma(I_i)$ lies completely in one set of the cover in X, which is what we set out to show.

■

Chapter 13
Fundamental Group: Capturing Holes

Theorem 13.2 Given topological spaces X and Y with $S \subset X$, homotopy relative to S is an equivalence relation on the set of all the functions from X to Y. In particular, if $S = \emptyset$, homotopy is an equvialence relation on the set of all continuous functions from X to Y.

Proof: Let f and g be continuous functions from X to Y. Let us denote $f \simeq_S g$ to mean that f and g are homotopic relative to $S \subset X$.

- **Reflexive:** Observe that this relation is reflexive. If $H(x,t) = f(x)$ for all $t \in [0,1]$, then it is trivial that H forms a homotopy relative to S between f and itself.

- **Symmetric:** If $f \simeq_S g$, then there is a continuous function $H : X \times [0,1] \to Y$ such that

$$\begin{aligned} H(x,0) &= f(x) && \text{for all } x \in X \\ H(x,1) &= g(x) && \text{for all } x \in X \\ H(x,t) = f(x) &= g(x) && \text{for all } x \in S, t \in [0,1] \end{aligned}$$

then consider $H(x, 1-t)$ and observe that

$$\begin{aligned} H(x,1) &= f(x) && \text{for all } x \in X \\ H(x,0) &= g(x) && \text{for all } x \in X \\ H(x,1-t) = f(x) &= g(x) && \text{for all } x \in S, t \in [0,1] \end{aligned}$$

is a homotopy that deforms g into f. Thus $g \simeq_S f$, so the relation is symmetric.

- **Transitive:** Now suppose $f \simeq_S g$ and $g \simeq_S h$. Then there exist continuous functions $H : X \times [0,1] \to Y$ and $G : X \times [0,1] \to Y$ such that

$$\begin{aligned} H(x,0) &= f(x) && \text{for all } x \in X \\ H(x,1) &= g(x) && \text{for all } x \in X \\ H(x,t) = f(x) &= g(x) && \text{for all } x \in S, t \in [0,1] \end{aligned}$$

and

$$\begin{aligned} G(x,0) &= g(x) && \text{for all } x \in X \\ G(x,1) &= h(x) && \text{for all } x \in X \\ G(x,t) = g(x) &= h(x) && \text{for all } x \in S, t \in [0,1]. \end{aligned}$$

Now observe that we can construct the function

$$F = \begin{cases} H(x,2t) & 0 \le t \le \frac{1}{2} \\ G(x,2t-1) & \frac{1}{2} < t \le 1 \end{cases}$$

which will be a homotopy relative to S from f to h. Note that continuity here is guaranteed by application of the pasting lemma, since H and G are continuous on the same intervals. In total, we then have that $f \simeq_S g$ and $g \simeq_S h$ imply $f \simeq_S h$, as desired.

■

Theorem 13.3 If α, α', β and β' are paths in a space X such that $\alpha \sim \alpha'$, $\beta \sim \beta'$ and $\alpha(0) = \beta(1)$, then $\alpha \cdot \beta \sim \alpha' \cdot \beta'$.

Proof: Since $\alpha \sim \alpha'$ and $\beta \sim \beta'$, there exists homotopies A and B which connect α to α' and β to β'. Consider the continuous function

$$H(x,t) = \begin{cases} A(2x,t) & 0 \le x \le \frac{1}{2} \\ B(2x-1,t) & \frac{1}{2} < x \le 1 \end{cases}$$

which is a homotopy from $\alpha \cdot \beta$ to $\alpha' \cdot \beta'$ since $H(x,0) = \alpha(x) \cdot \beta(x)$, $H(x,1) = \alpha'(x) \cdot \beta'(x)$, $H(0,t) = \alpha(0) = \alpha'(0)$ and $H(1,t) = \beta(0) = \beta'(0)$.

■

Theorem 13.4 Given paths α, β and γ where the following products are defined, then $(\alpha \cdot \beta) \cdot \gamma \sim (\beta \cdot \gamma) \cdot \alpha$ and $([\alpha] \cdot [\beta]) \cdot [\gamma] = [\alpha] \cdot ([\beta] \cdot [\gamma])$

Proof: Consider the homotopy given by

$$H(x,t) = \begin{cases} \alpha\left(\dfrac{4x}{2-t}\right) & 0 \le x \le \dfrac{2-t}{4} \\ \beta(4x+t-2) & \dfrac{2-t}{4} \le x \le \dfrac{3-t}{4} \\ \\ \gamma\left(\dfrac{4x-3+t}{1+t}\right) & \dfrac{3-t}{4} \le x \le 1. \end{cases}$$

Observe that

$$H(x,0) = \begin{cases} \alpha(2x) & 0 \le x \le \frac{1}{2} \\ \beta(4x-2) & \frac{1}{2} \le x \le \frac{3}{4} \\ \gamma(4x-3) & \frac{3}{4} \le x \le 1 \end{cases} = \alpha \cdot (\beta \cdot \gamma)$$

and

$$H(x,1) = \begin{cases} \alpha(2x) & 0 \le x \le \frac{1}{4} \\ \beta(4x-2) & \frac{1}{4} \le x \le \frac{1}{2} \\ \gamma(4x-3) & \frac{1}{2} \le x \le 1 \end{cases} = (\alpha \cdot \beta) \cdot \gamma.$$

Thus we see that H is continuous by the pasting lemma, $H(x,0) = \alpha \cdot (\beta \cdot \gamma)$ and $H(x,1) = (\alpha \cdot \beta) \cdot \gamma$. In addition, we see that $H(0,t) = \alpha \cdot (\beta \cdot \gamma)(0) = (\alpha \cdot \beta) \cdot \gamma(0)$ and $H(1,t) = (\alpha \cdot \beta) \cdot \gamma(1) = \alpha \cdot (\beta \cdot \gamma)(1)$. Thus we have that $\alpha \cdot (\beta \cdot \gamma) \sim (\alpha \cdot \beta) \cdot \gamma$, which implies that

$$[\alpha \cdot (\beta \cdot \gamma)] = [(\alpha \cdot \beta) \cdot \gamma]$$

as desired.

■

Theorem 13.5 Let α be a path with $\alpha(0) = x_0$. Then $\alpha \cdot \alpha^{-1} \sim e_{x_0}$, where e_{x_0} is the constant path at x_0.

Proof: Consider the homotopy

$$H(x,t) = \begin{cases} \alpha(2x) & 0 \le x \le \frac{1-t}{2} \\ \alpha^{-1}(2x-1) & \frac{1-t}{2} \le x \le 1-t \\ e_{x_0} & 1-t \le x \le 1. \end{cases}$$

which traverses α, α^{-1}, and then sits at x_0. Observe that

$$H(x,0) = \begin{cases} \alpha(2x) & 0 \le x \le \frac{1}{2} \\ \alpha^{-1}(2x-1) & \frac{1}{2} \le x \le 1. \end{cases} = \alpha \cdot \alpha^{-1}$$

while

$$H(x,1) = e_{x_0} \quad 0 \le x \le 1.$$

In addition, we have that $\alpha \cdot \alpha^{-1}(x) = e_{x_0}(x)$ for $x = 0, 1$. Also, H is continuous by the pasting lemma. Thus we have that

$$\alpha \cdot \alpha^{-1} \sim e_{x_0}$$

as desired. The proof is nearly identitical to show that $\alpha^{-1} \cdot \alpha \sim e_{x_0}$.

$\blacksquare$

Theorem 13.6 The fundamental group $\pi_1(X, x_0)$ is a group. The identity element is the class of homotopolically trivial loops based at x_0.

Proof:

Identity. With a group operation $\cdot$, we see that there is an identity element e_{x_0} such that $[\alpha] \cdot [\alpha^{-1}] = [\alpha^{-1}] \cdot [\alpha] = [e_{x_0}]$

Associativity. We have associativity of products by Theorem 13.4.

Inverse Elements. Inverse elements exist by simply defining $\alpha^{-1}(t) = \alpha(1-t)$. This will still be loop about x_0, and hence will continue to be a member of $\pi_1(X, x_0)$.

Closure. Finally, observe that the product is closed in the group, since any sequence of loops about x_0, their product

$$[\alpha_1] \cdot [\alpha_2] \cdot \ldots \cdot [\alpha_n] = [\alpha_1 \cdot \alpha_2 \cdot \ldots \cdot \alpha_n]$$

will itself be a loop about x_0, and hence by definition., an element which is already in the set. Thus the fundamental group $\pi_1(X, x_0)$ is in fact a group.

$\blacksquare$

Theorem 13.7 If X is path connected, then $\pi_1(X, p) \cong \pi_1(X, q)$ where $p, q \in X$.

Proof: We'll do this by constructing a bijective homomorphism. Since X is path connected, there must exist a path γ from p to q. Let α and β be loops centered at p, q respectively. Then observe that the function $\psi : \pi_1(X,p) \to \pi_1(X,q)$ defined as

$$\psi[\alpha] = \gamma^{-1}\alpha\gamma$$

is a homomorphism, since if $\alpha_1, \alpha_2 \in \pi_1(X,p)$,

$$\psi[\alpha_1\alpha_2] = \gamma^{-1}\alpha_1\alpha_2\gamma = \gamma^{-1}\alpha_1\gamma\gamma^{-1}\alpha_2\gamma = \psi[\alpha_1]\psi[\alpha_2].$$

We can similarly construct a homomorphism $\phi : \pi_1(X,q) \to \pi_1(X,p)$ as

$$\phi[\beta] = \gamma\beta\gamma^{-1}.$$

The proof is exactly the same as before: let $\beta_1, \beta_2 \in \pi_1(X,q)$. Then

$$\phi[\beta_1\beta_2] = \gamma\beta_1\beta_2\gamma^{-1} = \gamma\beta_1\gamma^{-1}\gamma\beta_2\gamma^{-1} = \phi[\beta_1]\phi[\beta_2].$$

Now observe that this homomorphism we constructed is in fact the inverse of ψ, since

$$\psi(\phi(\beta)) = \gamma^{-1}\phi(\beta)\gamma = \gamma^{-1}\gamma\beta\gamma^{-1}\gamma = \beta$$
$$\phi(\phi(\alpha)) = \gamma\phi(\alpha)\gamma^{-1} = \gamma\gamma^{-1}\alpha\gamma\gamma^{-1} = \alpha.$$

Therefore, ψ is a bijective homomorphism, which proves that $\pi_1(X,p) \cong \pi_1(X,q)$.

■

Corollay 13.8 Suppose X is a topological space and there is a path between the points p and q in X. Then $\pi_1(X,p)$ is isomorphic to $\pi_1(X,q)$.

Proof: Observe that this result is immediate since the proof of Theorem 13.7 relied on the fact that there exists a path between p and q. Thus the proof can be used exactly the same to show that path connectedness between two points is sufficient to guarantee that $\pi_1(X,p) \simeq \pi_1(X,q)$.

■

Exercise 13.9 Let α be a loop into a topological space X. Then $\alpha = \beta \circ \omega|_{[0,1]}$ where ω is the standard wrapping map and β is some continuous function from $\mathbb{S}^1$ into X. This relationship gives a correspondence between loops in X and continuous maps from $\mathbb{S}$ into X.

Solution: Consider the function $\omega^{-1} : \mathbb{S} \to [0,1]$ where $\omega(0) = \omega(1)$. As this is a continuous function, we then see that $\alpha \circ \omega^{-1} : \mathbb{S} \to X$ is a continuous function that maps out the curve X. Define this to be β. Then observe that we can write this as

$$\beta = \alpha \circ \omega^{-1} \implies \alpha = \beta \circ \omega$$

so that α can be written as a continuous from from $\mathbb{S} \to X$ composed with a continuous function from $[0,1] \to \mathbb{S}$, as desired.

□

Theorem 13.10 Let X be a topological space and let p be a point in X. Then a loop $\alpha = \beta \circ \omega|_{[0,1]}$ (where ω is the standard wrapping map and β is a continuous function from $\mathbb{S}^1$ into X) is homotopically trivial if and only if β can be extended to a continuous function from the unit disk $\mathbb{D}^2$ to X.

Proof:

■

Theorem 13.11 Show the following (1 denotes the trivial group):

1. $\pi_1([0,1]) \cong 1$
2. $\pi_1(\mathbb{R}^n) \cong 1$ for $n \geq 1$
3. $\pi_1(X) \cong 1$, if X is a convex set in $\mathbb{R}^n$
4. $\pi_1(X) \cong 1$ if X is a cone.
5. $\pi_1(X) \cong 1$ if X is a star-like space in $\mathbb{R}^n$ (a subset of $\mathbb{R}^n$ is called star-like if there is a fixed point $x_0 \in X$ such that for any $x \in X$, the line segment between x_0 and x lies in X; a five pointed star is an example of a star-like space that is not convex.)

Proof:

1. Observe that for any loop $\alpha \in [0,1]$, we can write a homotopy between $x_0 = \alpha(0)$ as
$$H(x,t) = tx_0 + (1-t)\alpha(x)$$
2. Again, loop $\alpha(\mathbf{x})$ based at $\mathbf{x_0}$ can be reduced to the trivial loop via the homotopy
$$H(\mathbf{x},t) = t\mathbf{x_0} + (1-t)\alpha(\mathbf{x})$$
3. A convex set in $\mathbb{R}^n$ has the property that every straight line between any two points in the set is entirely contained within the set. Thus we can apply the straight line homotopy to any loop based at x_0 as in (1.) and (2.).
4. Observe that every point on the cone can be connected to the apex via a straight line. Thus we can connect every loop based at the apex to itself via a straight-line homotopy.
5. In a star shaped figure, we can connect every point to one another via straight lines which intersect the fixed point x_0 without leaving the figure. Thus we can apply the straight line homotopy here as well.

■

Exercise 13.12 Show the following:

1. $\pi_1(\mathbb{S}^0, 1) \cong 1$ where $\mathbb{S}^0$ is the zero-dimensional sphere $\{-1,1\}$, the set of points unit distance from the origin in $\mathbb{R}^1$.
2. $\pi_1(\mathbb{S}^2) \cong 1$.
3. $\pi_1(\mathbb{S}^n) \cong 1$ for $n \geq 3$.

Solution:

1. In $\pi_1(\mathbb{S}^0, 1)$, the only element is the identity itself. Thus this group is literally trivial.
2. Consider a path γ in $\pi_1(\mathbb{S}^2)$, and suppose that γ is not a space filling curve. Then γ will miss at least a single point. Thus we can stereographically project γ on the sphere onto the $\mathbb{R}^2$ plane via a homeomorphism h.

   However, we know that any loop in $\mathbb{R}^2$ is homotopically trivial. Therefore there exists an a homotopy H from $h(\gamma)$ to the trivial loop. Now note that $h^{-1} \circ H$ will be a homeomorphism of γ to the trivial loop on $\mathbb{S}^2$. Thus $\pi_1(\mathbb{S}^2) = 1$.

   Now suppose γ is a space filling curve. Observe that via the Lebesgue number theorem, that this curve must enter and exit a finite number of times. Thus we can shift the curve over a particular point p in the open set, and do this a finite number of times. We can then stereographically project as before to shrink the curve on the surface to a point.

   Thus in either case, we see that any loop on $\mathbb{S}^2$ can be contracted to a single point via stereographic projection and the face that $\pi_1(\mathbb{R}^2) \cong 1$. Therefore we see that $\pi_1(\mathbb{S}^2) \cong 1$ as desired.

□

Exercise 13.13 Show that the cone over the Hawaiian earring is simply connected. Can you generalize your insight?

Solution: First observe that this space is path connected, since each ring of the Hawaiian earing are connected to the single point on the base of the cone and to the apex of the cone.

Now consider the unique point p on the base of the cone for which all rings intersect. Suppose α is a loop based at this point. With Theorem 10.25, we can deduce that α cannot traverse infinitely many rings in the Hawaiian earing. α is continuous and $[0, 1]$ is a compact interval and therefore it cannot be mapped into an infinitely long path, as this image would no longer be compact.

Thus any loop α based at p traverses a finite number of rings. Therefore, we can construct a homotopy H which lifts α over the apex of the cone and towards the point p itself, via a straight line homotopy (which we can do via the definition of the cone). Note that this will always be possible since there will only ever be a finite number of rings to lift over the apex.

Thus we have that $\pi(X, p) \cong 1$, but since this space is path connected we have that $\pi_1(X) \cong 1$. Therefore it is simply connected.

□

Theorem 13.14

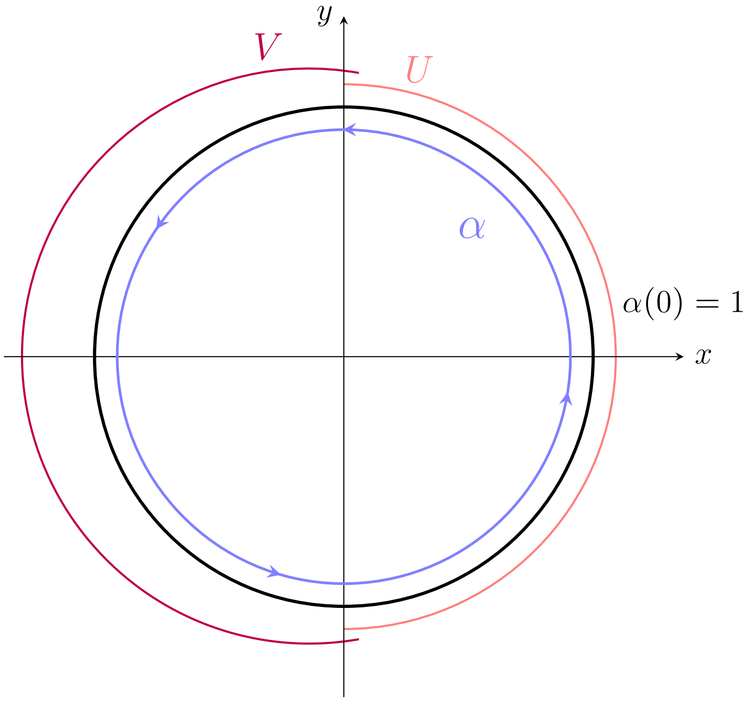
1. Any loop $\alpha : [0, 1] \to \mathbb{S}^1$ with $\alpha(0) = 1$ can be written $\alpha = \omega \circ \tilde{\alpha}$ where $\tilde{\alpha} : [0, 1] \to \mathbb{R}^1$ satisfies $\tilde{\alpha}(0) = 0$ and ω is the standard wrapping map.
2. If $\alpha : [0, 1] \to \mathbb{S}^1$ is a loop, then $\tilde{\alpha}(1)$ is an integer.
3. Loops α_1 and α_2 are equivalent in $\mathbb{S}^1$ if and only if $\tilde{\alpha_1}(1) = \tilde{\alpha_2}(1)$.
4. $\pi_1(\mathbb{S}^1) \cong \mathbb{Z}$.

Proof:

1. First observe that we can cover $\mathbb{S}^1$ with two open sets U and V, as demonstrated in the figure below. Since $[0,1]$ is a compact interval, and $\alpha[0,1] \to \mathbb{S}^1$, we know by Theorem 10.25 that we can divide $[0,1]$ into N intervals such that

$$\alpha\left(\left[\frac{i-1}{N}, \frac{i}{N}\right]\right)$$

lies in U or V.



Since U is not all of $\mathbb{S}^1$, we know that $\omega^{-1}(U)$ exists. If we define $\omega(0) = 1$ (i.e., if we specify that our rotation starts at 1) then $\omega^{-1}(U)$ will correspond to a union of open sets around every integer in $\mathbb{R}$. Also, $\omega^{-1}(V)$ will correspond to a union of intervals, each of which do not intersect any member of $\mathbb{Z}$.

Now since $\alpha\left(\left[\frac{i-1}{N}, \frac{i}{N}\right]\right) \subset U$ or V, we can map this image to $\mathbb{R}$ via ω^{-1}, starting from the first interval $\left[0, \frac{1}{N}\right]$ which is mapped to a neighborhood of 0 in $\mathbb{R}$. Thus we can define a function $\tilde{\alpha}$ as

$$\tilde{\alpha} = \omega^{-1} \circ \alpha$$

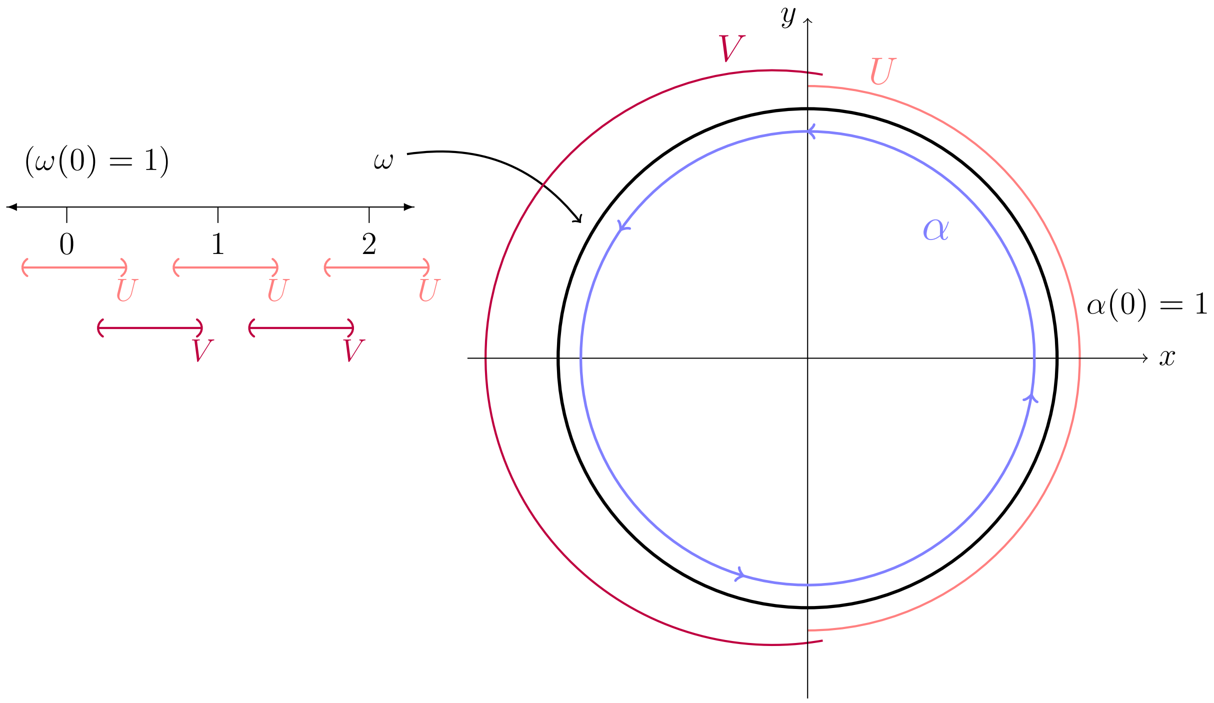
if we specify that $w(0) = 1$ and $\tilde{\alpha} : [0,1] \to \mathbb{R}$ by construction. Therefore, we can write

$$\alpha = \omega \circ \tilde{\alpha}$$

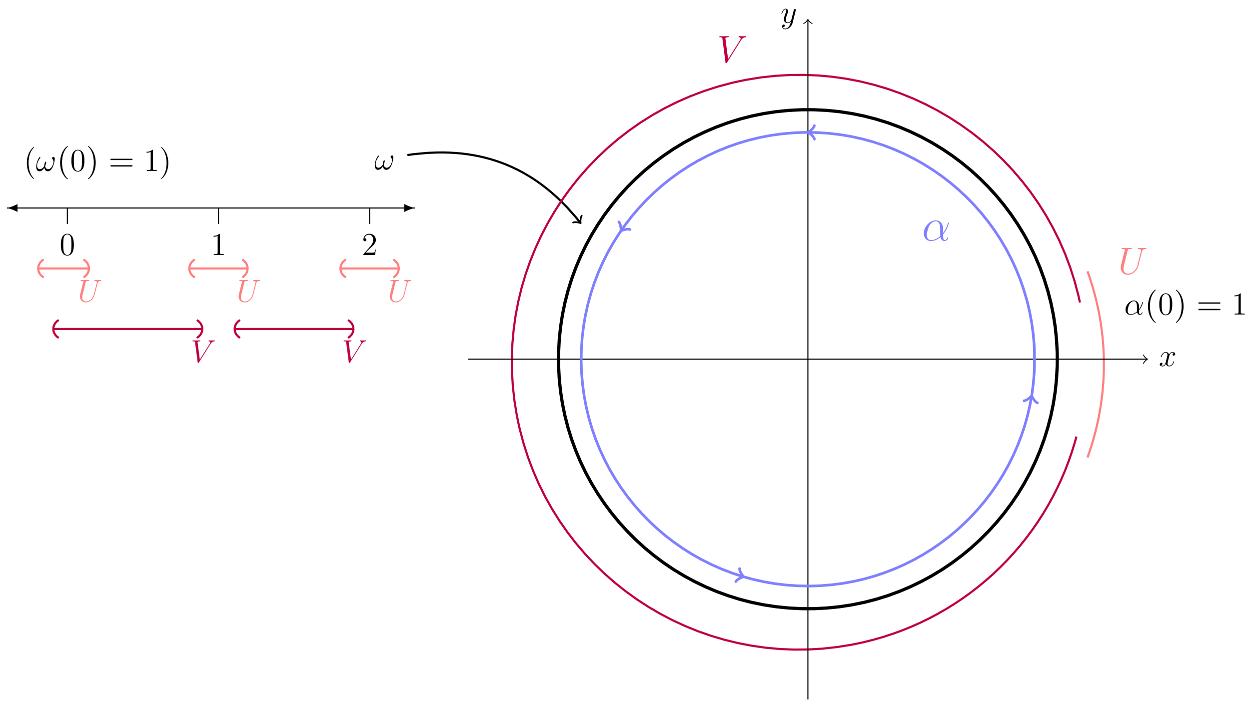
where $\alpha(0) = 0$.

2. Since $\alpha(1) = \alpha(0)$, we see that α must return to U at some point. And since there we can subdivide $[0,1]$ into finite intervals to keep track of the mapping, there will always be at most a finite number of rotations made around $\mathbb{S}$.

Now let us shrink U containing 1 in $\mathbb{S}$. As we shrink around 1, $\omega^{-1}(U)$ will still be a union of nieghborhoods of integers in $\mathbb{R}$. And since we are to free to shrink U, we see that the value of $al\tilde{p}ha$ must be an integer.



If $\tilde{\alpha}(1)$ is not an integer, then we can shrink U in $\mathbb{R}$ past this non-integer value. However, this implies that α is not a closed curve in $\mathbb{S}^1$ since $\alpha(1) \neq \alpha(0)$. Hence, $\omega^{-1} \circ \alpha = \tilde{\alpha}$ takes on integer values.



3. Suppose $\tilde{\alpha}_1(1) = \tilde{\alpha}_2(1)$. Note that $\tilde{\alpha_1}(0) = \widetilde{\alpha_2(}0)$, and as these are paths in $\mathbb{R}$ we can construct a stright line homotopy between the two paths relative to $\{0, 1\}$. Thus $\tilde{\alpha_1} \sim \tilde{\alpha_2}$, and there exists a homotopy H from $\tilde{\alpha_1}$ to $\tilde{\alpha_2}$.

   Since ω is continuous, $\omega \circ H$ is a continuous and a homotopy between α_1 and α_2, since (1) $(\omega \circ H)(0, t) = \omega \circ \tilde{\alpha_1} = \alpha_1$ and (2) $(\omega \circ H)(1, t) = \omega \circ \tilde{\alpha_2} = \alpha_2$, and the homotopy retains the endpoints.

4. Now consider the function $\gamma : \mathbb{S} \to \mathbb{Z}$ given by

$$\phi(\gamma) = \tilde{\gamma}(1).$$

Let $\gamma_1, \gamma_2 \in \pi_1(\mathbb{S}^1)$. Then observe that $\gamma_1 \cdot \gamma_2$ will be a path which comples $\tilde{\gamma}_1$ rotations, followed by $\tilde{\gamma}(2)$ rotations. Therefore

$$\phi(\gamma_1 \cdot \gamma_2) = \tilde{\gamma_1}(1) + \tilde{\gamma_2}(1) = \phi(\gamma_1) + \phi(\gamma_2).$$

Thus ϕ is a homomorphism. Now observe that part (3) of this problem proves injectivity, while surjectivity comes from the fact that for any $n \in \mathbb{Z}$, we can create a loop α such that α completes n rotations in $\mathbb{S}$, giving that $\tilde{\alpha}(1) = n$. Therefore ϕ is bijective and hence an isomorphism, so that $\pi_1(\mathbb{S}) \cong \pi_1(\mathbb{Z})$.

■

Theorem 13.15 Let $(X, x_0), (Y, y_0)$ be path connected spaces. Then

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$$

via the canonical map that takes a loop γ in $X \times Y$ to $(p \circ \gamma, q \circ \gamma)$ where $p : X \times Y \to X$ and $q : X \times Y \to Y$ are projection maps.

Proof: Observe that the map $(p \circ \gamma, q \circ \gamma)$ as defined above is a homomorphism. To show this, let $\gamma \in \pi_(X \times Y, (x_0, y_0))$. Then

$$\phi(\gamma) = (p \circ \gamma, q \circ \gamma) = (\gamma_x, \gamma_y) = (\gamma_x, e_{y_0}) \cdot (e_{x_0}, \gamma_y) = \phi(\gamma_x) \cdot \phi(\gamma_y)$$

where γ_x is a loop in X based at x_0, γ_y a loop in Y based at y_0. Observe that this is bijective since every loop in $\pi_1(X \times Y, (x_0, y_0))$ is mapped to a loop in $\pi_1(X, x_0) \times \pi_1(Y, y_0)$, and every loop in $\pi_1(X, x_0) \times \pi_1(Y, y_0)$ can be written as a loop in $\pi_1(X \times Y, (x_0, y_0))$. Therefore this is a isomorphism and thus $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$

■

Exercise 13.16 Find:

1. $\pi_1(X)$ where X is a solid torus.

2. $\pi_1(\mathbb{S}^2 \times \mathbb{S})$

3. $\pi_1(\mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2)$

4. $\pi_1(X)$, where X is a direct product of k_n copies of $\mathbb{S}^n$, with $k_n = 0$ for n sufficiently large.

Exercise 13.18 Check that for a continuous function $f : X \to Y$, the induced homomorphism f_* is well-defined (that is, the image of an equivalence class is independent of the chosen representative.) Show that it is indeed a group homomorphism.

Solution: Observe that since $\alpha \sim \beta$, there exists a homotopy H between the two paths that fixes the endpoints. Then observe that $f \circ H$ is (1) a continuous function and (2) a homotopy from $f(\alpha)$ to $f(\beta)$. Thus $f(\alpha) \sim f(\beta)$. Therefore, if $\alpha \sim \beta$ then $f(\alpha) \sim f(\beta) \implies [f \circ \alpha] = [f \circ \beta] \implies f_*([\alpha]) = f_*([\beta])$ so that our definition is well defined.

To show it is a group homomorphism, observe that

$$f_*([\alpha \cdot \beta]) = [f \circ (\alpha \cdot \beta)] = [f \circ \alpha \cdot f \circ \beta] = [f \circ \alpha] \cdot [f \circ \beta] = f_*([\alpha]) \cdot f_*([\beta]).$$

Thus we see that this forms a group homomorphism.

□

Theorem 13.19 The following are true:

1. If $f : (X, x_0) \to (Y, y_0)$ and $g : (Y, y_0) \to (Z, z_0)$ are continuous maps, then $(g \circ f)_* = g_* f_*$.
2. If $\mathrm{id} : (X, x_0) \to (Y, y_0)$ is the identity map, then $\mathrm{id}_* : \pi_1(X, x_0) \to \pi_1(X, x_0)$ is the identity homomorphism.

Proof:

1. Let $[\alpha] \in \pi_1(X, x_0)$. Then

   $$(g \circ f)_*([\alpha]) = [(g \circ f) \circ \alpha] = [g \circ (f \circ \alpha)] = g_*([f_*([\alpha])]) = g_* \circ f_*([\alpha])$$

   so that $(g \circ f)_* = g_* f_*$.

2. Let $[\alpha] \in \pi_1(X, x_0)$. Then

   $$id_*([\alpha]) = [id \circ \alpha] = [\alpha].$$

   Since this is a homomorphism on the group which sends every group element to itself, we have that this is an identity homomorphism.

■

Theorem 13.20 If $h : X \to Y$ is a homeomorphism then

$$h_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$$

is a group isomorphism. Thus homeomorphic, path-connected spaces have isomorphic fundamental groups.

Proof: If $h : X \to Y$ is a homeomorphism, then h is continuous and bijective. Therefore, if $\alpha \in \pi_1(X, x_0)$, then $h_* = [h \circ \alpha]$. Since h^{-1} exists and is continuous, we then now if $\beta \in \pi_1(Y, y_0)$ then $h_*^{-1} = [h^{-1} \circ \alpha]$ is also a homomorphism. Now observe that

$$h_*[h_*^{-1}[\beta]] = [h \circ [h^{-1} \circ \beta]] = [\beta]$$
$$h_*^{-1}[h_*[\alpha]] = [h^{-1} \circ [h \circ \alpha]] = [\alpha].$$

Therefore, we see that h_*^{-1} is the inverse homomorphism of h_*, which implies that the homomorphism is bijective. Therefore, the two groups are isomorphic.

■

Theorem 13.22 If $f, g : (X, x_0) \to (Y, y_0)$ are continuous functions and f is homotopic to g relative to x_0, then $f_* = g_*$.

Proof: Since $f \simeq g$, there exists a homotopy H such that $H(x, 0) = f(x), H(x, 1) = g(x)$ and $H(x_0, t) = f(x_0) = g(x_0)$ for all $t \in [0, 1]$. Let $\alpha \in \pi_1(X, x_0)$. Then observe that $H(\alpha(x), t)$ is (1) continuous and (2) a homotopy from $f \circ \alpha$ to $g \circ \alpha$. Therefore

$$f \circ \alpha \simeq g \circ \alpha \implies [f \circ \alpha] = [g \circ \alpha] \implies f_* = g_*.$$

■

Lemma 13.23 Homotopy equivalence of spaces is an equivalence relation.

Proof: We can show that this satisfies the axioms for an equivalence relation.

Reflexive. Observe that if we let $f = g = id_X$, then $g \circ f = id_X$ and $f \circ g = id_X$. Thus a topological space X is homotopy equivalent to itself.

Symmetric. The defintion of homotopy equivalence makes this obvious. Suppose X is homotopy equivalent to Y. By definition, there exists continuous maps $f : X \to Y$ and $g : Y \to X$ such that

$$g \circ f \simeq id_X \qquad f \circ g \simeq id_Y.$$

Thus Y is homotopy equvalent to X since there exists continuous functions $g : Y \to X$ and $f : X \to Y$ such that

$$f \circ g \simeq id_Y \qquad g \circ f \simeq id_X.$$

Transitive. Suppose X is homotopy equivalent to Y which is homotopy equivalent to Z. By definition, there exist continuous functions $f_1 : X \to Y$, $f_2 : Y \to X$, $g_1 : Y \to Z$, $g_2 : Z \to Y$ such that

$$f_1 \circ f_2 \simeq id_X \qquad f_2 \circ f_1 \simeq id_Y \text{ and } g_1 \circ g_2 \simeq id_Y \qquad g_2 \circ g_1 \simeq id_Z$$

■

Theorem 13.24 If $f : X \to Y$ is a homotopy equivalence and $y_0 = f(x_0)$, then $f_* : \pi_1(X, x_0) \to \pi_1(Y, y_0)$ is an isomorphism. In particular, if $X \sim Y$, then $\pi_1(X) \cong \pi_1(Y)$.

Proof: Since f is continuous from X to Y, we already have a homomorphism from $\pi_1(X, x_0) \to \pi_1(Y, y_0)$, given by f_*. Now consider the induced homomorphism g_* where

$$g \circ f \simeq id_X \quad \text{and} \quad f \circ g \simeq id_Y.$$

Then observe that if $\alpha \in \pi_1(X, x_0)$ then

$$g_* \circ f_*[\alpha] = (g_* \circ f_*)[\alpha] = [g \circ f \circ \alpha].$$

Now since $g \circ f \simeq id_X$, we know that there exists a homotopy H such that $H(x, 0) = g \circ f$ and $H(x, 1) = id_X$ while $H(x_0, t) = g \circ f(x_0) = id_X(x_0) = x_0$. Then observe that $H(\alpha(x), t)$ is a homotopy from $g \circ f \circ \alpha$ to α. As these two paths are homotopic, their equivalence classes should be the same. Therefore, we see that

$$[g \circ f \circ \alpha] = [\alpha].$$

Now if $\beta \in \pi_1(Y, y_0)$

$$f_* \circ g_*[\beta] = (f_* \circ g_*)[\beta] = [f \circ g \circ \beta] = [\beta]$$

By the same argument. Therefore, g_* is an inverse homomorphism of f_*, so that f_* is ultimately an isomorphism between the two groups. Thus $\pi_1(X, x_0) \cong \pi_1(Y, y_0)$.

■

Exercise 13.25 Show that for $n \geq 0$, $\mathbb{R}^{n+1} - \{0\}$ can be strong deformation retracted onto $\mathbb{S}^n$

Solution: Consider the homotopy $R : \mathbb{R}^{n+1} \to \mathbb{S}^n$ such that

$$\begin{aligned} R(\mathbf{x}, 0) = \mathbf{x} &\quad \text{for all } \mathbf{x} \in \mathbb{R}^{n+1} \\ R(x, 1) = r(x) &\quad \text{for all } \mathbf{x} \in \mathbb{R} \\ R(a, t) = a &\quad \text{for all } a \in \mathbb{S}^n \end{aligned}$$

where

$$r(\mathbf{x}) = \frac{\mathbf{x}}{||\mathbf{x}||}.$$

Thus we see that $\mathbb{R}^{n+1} - \{0\}$ can be strong deformation retracted onto $\mathbb{S}^n$.

□

Lemma 13.26 If A is strong deformation retract of X, then A and X are homotopy equivalent.

Proof: Since A is a strong deformation retract of X, we know that there exists a continuous function $r : X \to A$ such that $r(a) = a$ for all $a \in A$. Consider also the inclusion map $i : A \to X$. Observe that

$$i \circ r : X \to X \quad r \circ i : A \to A$$

and

$$r \circ i = id_A \quad i \circ r \simeq id_X.$$

Thus by defintion we see tha $A \sim X$.

■

Theorem 13.27 $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$, for any $n \neq 2$.

Proof: Suppose for contradiction that $\mathbb{R}^2$ is homeomorphic to $\mathbb{R}^n$. Then if we poke a hole in $\mathbb{R}^2$, we can embed a circle in the space such that its interior contains the hole. We can then strong deformation retract $\mathbb{R}^2$ onto the boundary of the disk. We then see the fundamental group is $\mathbb{Z}$ by Theorem 13.24.

However, we know that $\mathbb{R}^n$, $n \geq 2$ with one hole missing is still a trivial group. In $\mathbb{R}^n$, we can move around the circle we embdedded in $\mathbb{R}^2$ to still compute a trivial fundamental group. But this is a contradiction, since these two spaces are said to be homeomorphic but their fundamental groups are inconsistent under change. Therefore, $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^n$ for any $n \geq 2$.

Also observe that $\mathbb{R}^2$ is not homemorphic to $\mathbb{R}$. This is because $\pi_1(\mathbb{R}^2) \cong \mathbb{R} \times \mathbb{R}$. Also, $\mathbb{R}^2$ is not homeomorphic to $\mathbb{R}^0$, as this is a single point. If we delete this point the fundamental group is empty, while the fundamental group of $\mathbb{R}^2$ would become equivalent to $\mathbb{Z}$, and hence these two spaces are not homemorphic. Therefore, we see that $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^n$ for any $n \neq 2$.

■

Exercise 13.28 Let x and y be two points in $\mathbb{R}^2$. Show that $\mathbb{R}^2 - \{x, y\}$ strong deformation retracts onto the figure eight. In addition, show that $\mathbb{R}^2$. Show that $\mathbb{R}^2 - \{x, y\}$ strong deformation retracts onto a theta space.

Solution: Observe that the figure eight and the theta space both have two holes inside of them. If we configure both of these holds to individually contain x and y, then we can retract $\mathbb{R}$ ont the boundaries of the figure eight and theta space.

Note we would not be able to do this without first poking two holes in $\mathbb{R}$, since we would otherwise not be able to retract the interior of the each hole in the figure eight or theta space to its boundaries (for the same reason we can't retract $\mathbb{D}^2$ to its boundary).

□

Theorem 13.29 If $r : X \to A$ is a strong deformation retraction and $a \in A$, then $\pi_1(X, a) \cong \pi_1(A, a)$.

Proof: Suppose $r : X \to A$ is a strong deformation retraction. Then by Lemma 13.26, we know that A and X are homotopy equivalent. Moreover, by Theorem 13.24, we have that $\pi_1(X, a) \cong \pi_1(A, a)$ for $a \in A$.

■

Exercise 13.30 Calculate the fundamental group of the following spaces.

1. An annulus.
2. A cylinder.
3. The Möbius Band.
4. An open 3-ball with a diameter removed.

Solution:

1. Suppose that we embed an annulus at the origin of the complex plane. Then it is given by $\{z : R_1 < |z| < R_2\}$ where $R_1 < R_2$. It should be fairly obvious that we can construct a strong deformation retract to the set of points $\{z : |z| = R_1\}$; that is, to the inner circle of the annulus. As the fundamental group of the circle is $\mathbb{Z}$, we can thus conclude that the fundamental group of the annulus is also $\mathbb{Z}$ by Theorem 13.29.

2. For a cylinder, we can embedd such a structure in $\mathbb{R}^3$, which in this space we can construct a strong deformation retract between the set of points on the cylinder to one of the two disks which define the top and the bottom of the cylinder. Since the fundamental group of a disk is trivial, we see that the fundamental group of the cylinder must also be trivial by Theorem 13.29.

3. Observe that if we take a Möbius band and strong deformation retract the set of points to one of its boundaries, we'll get a closed curve, which we can then form a strong deformation retraction to a circle. Since the fundamental group of a circle is $\mathbb{Z}$, we see that the fundamental group of the Möbius band is also $\mathbb{Z}$.

   However, if the cylinder does not have a filled top (i.e., the cylinder is just a piece of paper folded on its ends) then the group is trivial

4. Observe that we can strong deformation retract an open 3-ball with a diameter removed to a circle with a hole removed from its center. Since this has a fundamental group of $\mathbb{Z}$, we see that by Theorem 13.27 that the open 3-ball with diameter removed also has a fundamental group of $\mathbb{Z}$.

□

Theorem 13.32 Let A be a retract of X via the inclusion $i : A \hookrightarrow X$ and retraction $r : X \hookrightarrow A$. Then for $a \in A$, $i_* : \pi_1(A, a) \to \pi_1(X, a)$ is injective and $r_* : \pi_1(X, a) \to \pi_1(A, a)$ is surjective.

Proof: First note that for any $\alpha \in \pi_1(A, a)$ we have that

$$r_* \circ i_*([\alpha]) = [r \circ i \circ \alpha] = [\alpha].$$

Therefore, we see that $r_* \circ i_*([\alpha]) = id_*$ is the identity homomorphism on $\pi_1(A, a)$. Now suppose that i_* is not injective. Then we'll have that $r_* \circ i_* \neq id_*$, which is a contradiction. Furthermore, if r_* is not surjective then $r_* \circ i_* \neq id_*$. Thus we see that i_* is injective and r_* is surjective.

■

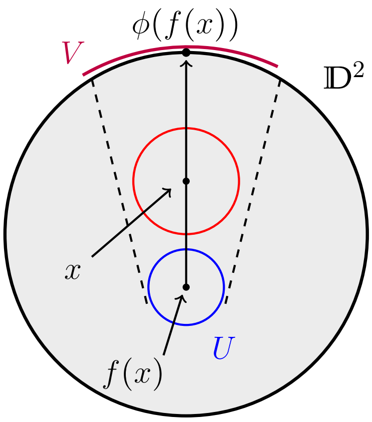
Theorem 13.33 (No retraction theorem for $\mathbb{D}^2$.) There is no retraction from $\mathbb{D}^2$ to its boundary.

Proof: Suppose for a contradiction that there exists a retraction $r : \mathbb{D}^2 \to \mathbb{S}^1$. Then the inclusion map $i_* : \pi_1(\mathbb{S}^1) \to \pi_1(\mathbb{D}^2)$ should be injective, this is impossible since $\pi_1(\mathbb{S}^1) = \mathbb{Z}$ while $\pi_1(\mathbb{D}^2)$ is trivial. Thus there is no such r.

■

Theorem 13.34 (Brouwer Fixed Point Theorem for $\mathbb{D}^2$.) Let $f : \mathbb{D}^2 \to \mathbb{D}^2$ be a continuous map. Then there is some $x \in \mathbb{D}^2$ for which $f(x) = x$.

Proof: Suppose for a contradiction that there exists a continuous function $f : \mathbb{D}^2 \to \mathbb{D}^2$ such that $f(x) \neq x$ for all $x \in \mathbb{D}^2$. Consider the retraction $\phi(x) = x$ if $x \in \mathbb{S}^1$ and $\phi(x) = f(x)|_{\text{proj}}$ where $f(x)_{\text{proj}}$ is the projection of $f(x)$ to its boundary through the straightline between $f(x)$ and x.



Observe that this function is continuous, since for any open set containing $V \subset \mathbb{S}^1$ containing $\phi(f(x))$ there exists an open set $U \in \mathbb{D}^2$ containing $f(x)$ such that $\phi(U) \subset V$. See the figure.

Note that what we have is a continuous retraction of $\mathbb{D}^2$ to its boundary, $\mathbb{R}$. However, we know from Theorem 13.33 that this is a contradiction. Thus there cannot be any such f, so that for a continuous $f : \mathbb{D}^2 \to \mathbb{D}^2$ there must exist an $x \in \mathbb{D}^2$ such that $f(x) = x$, as desired.

■

Theorem 13.40 Let $X = U \cup V$, where U and V are open and path connected and $U \cap V$ is path-connected, simply connected and nonempty. Then $\pi_1(X) \cong \pi_1(U) \star \pi_1(V)$.

Proof: Let $x_0 \in U \cap V$ and let Γ be a loop based at x_0. Then observe that $\Gamma : [0, 1] \to X$ is a path from a compact interval, and U and V form an open cover of X. By Theorem 10.25 we may divide the interval into finite subintervals such that the image of each interval lies in U or V.

We now claim that we can write any loop Γ as product of loops in $\pi_1(U)$ and $\pi_1(V)$. Since $U \cap V$ is path connected, we know that for every every time our path intersects $U \cap V$ there exists a point $p \in U \cap V$ such that we can glue a new path γ from p to x_0. This path will either lie entirely in U or in V, thus becoming a member of $\pi_1(U)$ and $\pi_1(V)$. And by Theorem 10.25, this can be done a finite number of times.

Now consider the function

$$\phi(\Gamma) = \alpha_1\beta_1 \cdots \alpha_n\beta_n$$

where $\alpha_1\beta_1 \cdots \alpha_n\beta_n$ is the decomposition of Γ and $\alpha_i \in \pi_1(U)$ while $\beta_i \in \pi_i(V)$ for $i = 1, 2, \ldots, n$. We'll now show this is a homomorphism. For any two paths Γ_1, Γ_2, each have some decomposition $\alpha_1^{(1)}\beta_1^{(1)} \cdots \alpha_n^{(1)}\beta_n^{(1)}$ and $\Gamma_2 = \alpha_1^{(2)}\beta_1^{(2)} \cdots \alpha_n^{(2)}\beta_n^{(2)}$. Therefore, we see that

$$\Gamma_1 \cdot \Gamma_2 = \alpha_1^{(1)}\beta_1^{(1)} \cdots \alpha_n^{(1)}\beta_n^{(1)}\alpha_1^{(2)}\beta_1^{(2)} \cdots \alpha_n^{(2)}\beta_n^{(2)}$$

so that

$$\phi(\Gamma_1 \cdot \Gamma_2) = \alpha_1^{(1)}\beta_1^{(1)} \cdots \alpha_n^{(1)}\beta_n^{(1)}\alpha_1^{(2)}\beta_1^{(2)} \cdots \alpha_n^{(2)}\beta_n^{(2)} = \phi(\Gamma_1)\phi(\Gamma_2).$$

Thus observe that for any $\alpha_1\beta_1 \cdots \alpha_n\beta_n \in \pi_1(X)$, this corresponds to some unique path $\Gamma \in pi_1(X)$ such that

$$\phi(\Gamma) = \alpha_1\beta_1 \cdots \alpha_n\beta_n$$

Thus we get surjectivity for free, since any member of $\pi_1(U) \star \pi_1(V)$ corresponds to some path $\Gamma \in \pi_1(X)$; therefore, the image of Γ under ϕ is then the element of $\pi_1(U) \star \pi_1(V)$ we began with. However, this also lends uniqueness, since every member of $\pi_1(U) \star \pi_1(V)$ corresponds uniquely to some path of $\Gamma \in \pi_1(X)$. Therefore we see that this is a isomorphism, so that $\pi_1(X) \cong \pi_1(U) * \pi_1(V)$.

■

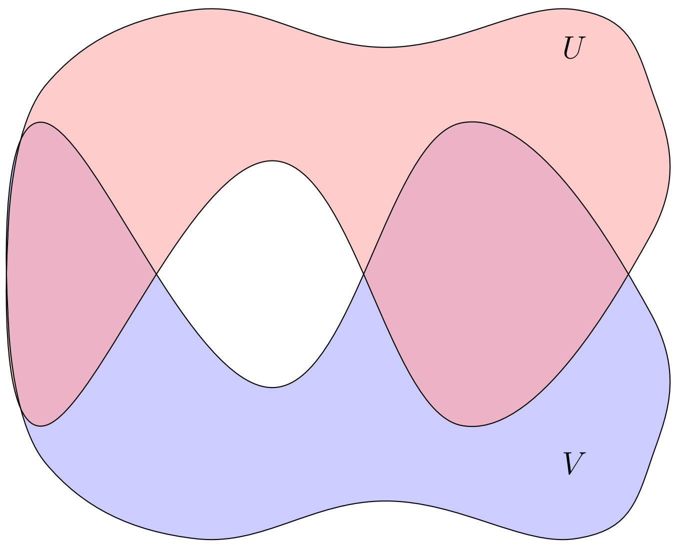
Exercise 13.41 Let X be the bouqeuet of n circles. What is $\pi_1(X)$?

Solution: The bouquet of n circles simply identifies a point on a set of n circles to the same point. Thus we see by repeated application of Theorem 13.30, $\pi_1(X) \cong \pi_1(\mathbb{S}) * \pi_1(\mathbb{S}) * \cdots * \pi_1(\mathbb{S}) = \mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z}$. That is, the free product of n groups of $\mathbb{Z}$.

□

Exercise 13.32 Find a path-connected space X with open, path connected subsets U and V of X such that $X = U \cup V$ where U and V are both simply connected, but X is not simply connected. Conclude that the hypothesis that $U \cap V$ is path connected is necessary.

Solution: Consider the sets



where we see that $U \cap V$ is not path connected. In this example we see that the consequence of this is that the union of the sets $U \cup V$ is no longer simply connected, and hence it has a nontrivial fundamental group. However, if we were to ignore the condition that $U \cap V$ be path connected, then Van Kampen's theorem in this case would otherwise guarantee that its fundamental group should be the free product of two trivial groups, and hence be a trivial group itself. Thus path connectedness of $U \cap V$ is a necessary condition for Van Kampen's theorem to be true.

□

Theorem 13.44 Let X be a wedge of two cones over two Hawaiian earrings where they are identified at the points of tangency of the circles of each Hawaiian earring, as in Figure 13.9. Then $\pi_1(X) \ncong 1$.

Proof: Suppose α_n is a loop on the n-th ring on the left Hawaiian earring, while β_n is a loop on the n-th ring on the right Hawaiian earing. Thus consider the path

$$\gamma = \alpha_1\beta_1\alpha_2\beta_2 \cdots \alpha_k\beta_k$$

where $k \in \mathbb{N}$. Observe that $\gamma \in \pi_1(X)$ (more specifically, its equivalence class is a member). However, if we attempt to lift this path towards the apex of the cone, which we can individually do without any issue for $\alpha_1\alpha_2 \cdots \alpha_k$ and $\beta_1\beta_2 \cdots \beta_k$, we run into an issue as $k \to \infty$. This is because $[0, 1]$ is a compact interval and cannot be mapped into an infinite path, so that if we attempted to form any homotopy it would automatically fail to be compact and hence continuous if we try to lift the path γ up simultaneously. Thus this path is not homotopic to a point, so that $\pi_1(X) \neq 1$.

■